THE UNIVERSAL QUANTUM INVARIANT AND COLORED IDEAL TRIANGULATIONS

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Abstract. The Drinfeld double of a finite dimensional Hopf algebra is a quasi-triangular Hopf algebra with the canonical element as the universal R-matrix, and one can obtain a ribbon Hopf algebra by adding the ribbon element. The universal quantum invariant of framed links is constructed using a ribbon Hopf algebra. In that construction, a copy of the universal R-matrix is attached to each crossing, and invariance under the Reidemeister III move is shown by the quantum Yang-Baxter equation of the universal R-matrix. On the other hand, R. Kashaev showed that the Heisenberg double of a finite dimensional Hopf algebra has the canonical element (the *S-tensor*) satisfying the pentagon relation. In this paper we reconstruct the universal quantum invariant using the Heisenberg double, and extend it to an invariant for *colored singular triangulations* of topological spaces, especially for *colored ideal triangulations* of tangle complements. In this construction, a copy of the S-tensor is attached to each tetrahedron, and invariance under the *colored Pachner* $(2,3)$ *moves* is shown by the pentagon relation of the S-tensor.

Contents

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1. Introduction

The universal quantum invariant [Law89, Law90, Oht93] associated to a ribbon Hopf algebra is an invariant of framed tangles in a cube which has the universal property over Reshetikhin-Turaev invariants [RT90]. The relationship between the universal quantum invariant and 3-dimensional, global, topological properties of tangles is not well understood, mainly because of the 2-dimensional definition using link diagrams. In this paper, we give a reconstruction of the universal quantum invariant using *colored ideal triangulations* of tangle complements, and give an extension of the universal quantum invariant to an invariant for *colored singular triangulations* of topological spaces. We expect that our framework will become a new method to study the quantum invariants in a 3-dimensional way.

1.1. Reconstruction and extension of the universal quantum invariant.

In the theory of quantum groups there are two doubles of a finite dimensional Hopf algebra A. One is the *Drinfeld double* $D(A)$ and the other is the *Heisenberg double* $H(A)$. They are both isomorphic to $A^* \otimes A$ as vector spaces.

The Drinfeld double $D(A)$ is a quasi-triangular Hopf algebra with a canonical element $R \in D(A)^{\otimes 2}$ as the universal R-matrix, which satisfies the quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

see e.g. [Dri87, Maj98, Maj99]. One can obtain a ribbon Hopf algebra $D(A)^\theta$ by adding the ribbon element θ. In what follows we assume that the universal quantum invariant is associated to $D(A)^\theta$ for a finite dimensional Hopf algebra A.

The Heisenberg double $H(A)$ is a generalization of the Heisenberg algebras [Sem92, Lu94, Kap98]. Kashaev [Kas97] showed that a canonical element $S \in H(A)^{\otimes 2}$, which we call the *S-tensor*, satisfies the pentagon relation

$$S_{12}S_{13}S_{23} = S_{23}S_{12}.$$

Kashaev also constructed an algebra embedding $\phi\colon D(A) \to H(A) \otimes H(A)^{\mathrm{op}}$ such that the image of the universal R-matrix is a product of four variants of the S-tensor:

$$\phi^{\otimes 2}(R) = S''_{14}S_{13}\tilde{S}_{24}S'_{23} \quad \in \left(H(A) \otimes H(A)^{\mathrm{op}}\right)^{\otimes 2}, \tag{1.1}$$

where S', S'' and $\tilde{S}$ are the images of S by maps constructed from the antipode, see Theorem 3.5.

The situation (1.1) reminds us the situation of an *octahedral triangulation* [CKK14, Yok11, We05] of the complement of a link in $S^3 \setminus \{\pm\infty\}$, where an octahedron consisting of four tetrahedra is associated to each crossing of a link diagram. [1] Because

[1]Throughout this paper we consider only topological ideal triangulations and we do not consider geometric structures on them.

of this similarity, it is natural to ask if we can reconstruct the universal quantum invariant of a tangle T using a octahedral triangulation of the complement of T in a cube, where a copy of the S-tensor is associated to each tetrahedron in the octahedral triangulation.

The answer is yes, and in this paper we give such a reconstruction. Here, we would like to stress that, we can construct the universal quantum invariant using the S-tensor by simply rewriting the universal R-matrix by (variants of) the S-tensor using $\phi^{\otimes 2}$. However, an important result is that we give a way to relate a copy of the S-tensor to an ideal tetrahedron in an octahedral triangulation, and a way to read these copies of the S-tensor to obtain the universal quantum invariant. The framework of the above reconstruction enables us to extend the universal quantum invariant to an invariant for *colored singular triangulations* of topological spaces, which are singular triangulations with an additional structure (coloring).

1.2. **Universal quantum invariant as a state sum invariant with weights in a non-commutative ring.** Let us explain the nature of the coloring on a singular triangulation from a viewpoint of state sum constructions.

One can obtain a state sum invariant of tangles and 3-manifolds by associating a 6*j-symbol* to each tetrahedron in a triangulation of a 3-manifold, where the values of the 6j-symbol on colors on the edges of a tetrahedron give a weight of the state sum [TV92, Oc94].

In the context of hyperbolic geometry, there are several attempts to construct a state sum invariant of hyperbolic links and hyperbolic 3-manifolds, such that, to each tetrahedron one associates Faddeev and Kashaev's *quantum dilogarithm* [FK94], and the values of them on the cross ratio moduli of hyperbolic ideal triangulation give weights of the state sum. The first discovery of such an invariant seems to be [Kas94] by Kashaev. For an odd integer $N > 1$, he proposed an invariant K_N for pairs (M, L) of a closed oriented 3-manifold M and a link L in M, using the cyclic 6j-symbol $R(p, q, r)$. He also showed that $R(p, q, r)$ is obtained from certain operators S and $\Psi_{p,q,r}$ on $\mathbb{C}^N \otimes \mathbb{C}^N$, where S satisfies a certain pentagon relation and $\Psi_{p,q,r}$ satisfies the quantum dilogarithm identity. Here, a classical limit of Faddeev and Kashaev's quantum dilogarithm identity gives Rogers's identity for Euler's dilogarithm, and this fact seems to lead Kashaev to his famous conjecture about the relationship between his invariant and the hyperbolic volumes of link complements [Kas97'].

Murakami and Murakami [MM01] showed that Kashaev's R-matrix is conjugate (up to scalar multiplication) to that of the *colored Jones polynomial* J_N with $q = \exp\frac{2\pi i}{N}$ and an N-dimensional irreducible representation of $U_q(sl_2)$. This result also showed the well-definedness of Kashaev's invariant. Murakami-Murakami's construction could be seen as a state sum invariant with a weight associated to a crossing, consisting of four quantum dilogarithms.

Baseilhac and Benedetti [BB04, BB05, BB07, BB11] obtained Kashaev's R-matrix in their settings constructing *quantum hyperbolic invariants* $H_N(M, L, \rho)$ for triples (M, L, ρ) of a compact oriented 3-manifold M, non-empty link L in M, and ρ a flat principal bundle over M with structure group $PSL(2, \mathbb{C})$. They associate a quantum dilogarithm to a tetrahedron in a singular triangulation, and for the proof of well-definedness they faced the *symmetrization* problem and the *global* problem for the values of the quantum dilogarithm on the cross ratio moduli.

Note that each state sum appearing above has weights in $\mathbb{C}$. On the other hand, the universal quantum invariant could be seen as a state sum invariant with weights taking values in a non-commutative ring (ribbon Hopf algebra); a weight is associated to each fundamental tangle (see Figure 2.2), especially a copy of the universal R-matrix is associated to each crossing, and one takes products of the

weights in the order following the orientations of strands of a tangle (see Section 2.2 for the precise definition). We would like to apply this framework to a state sum construction using (singular) triangulations, i.e., our motto is:

Using an element S satisfying a pentagon relation in a non-commutative ring, construct a state sum invariant of 3-manifolds by associating a copy of S to each tetrahedron of a triangulation.

Namely, we would like to construct a state sum invariant using a 6j-symbol (resp. quantum dilogarithm) and its pentagon identity (resp. quantum dilogarithm identity) treated as a function on colors on edges of tetrahedra (resp. cross ratio moduli of ideal tetrahedra) and its pentagon relation, respectively. Then one does not need to fix colors on the edges of a tetrahedron or cross ratio modulus of an ideal tetrahedron, and for the proof of well-defindness, instead of the pentagon identity of a 6j-symbol or of the symmetrization and the global problem for a quantum dilogarithm, one would work with an algebraic pentagon relation. Moreover, we expect that such an invariant involves combinatorial information of a triangulation in its non-commutative algebra structure, including the consistency and the completeness conditions of ideal triangulations when we fix cross ratio moduli.

However, when we use a (singular) triangulation, we do not have a canonical order on the set of weights on tetrahedra in the triangulation. Thus we need to fix an order, then we naturally come to a notion of a *colored singular triangulation*: each tetrahedron is sticked by two colored strands and strands are connected globally in the triangulation. Then a copy of the S-tensor is associated to the two strands of each tetrahedron and we can read the copies of the S-tensor in the order following the orientations of the colored strands. Corresponding to the *Pachner* $(2,3)$ *move* and the $(0,2)$ *move* of singular triangulations, we define *colored Pachner* $(2,3)$ *moves* and *colored* $(0,2)$ *moves* of colored singular triangulations. The extension of the universal quantum invariant is an invariant of colored singular triangulations up to certain *colored moves*. In this paper, following the order of the discovery, these colored strands first arise from a tangle diagram. Then we consider colored strands more generally in singular triangulations of topological spaces.

1.3. **Organization of this paper.** Section 2 is devoted to the definition of the universal quantum invariant associated to a ribbon Hopf algebra. In Section 3, we recall the Drinfeld double $D(A)$ and the Heisenberg double $H(A)$ of a finite dimensional Hopf algebra A, where the universal R-matrix in $D(A)^{\otimes 2}$ and the S-tensor in $H(A)^{\otimes 2}$ satisfy the quantum Yang-Baxter equation and the pentagon equation, respectively. We also recall from [Kas97] how there elements are related via the algebra embedding of $D(A)$ into $H(A)\otimes H(A)^{\mathrm{op}}$. In Section 4, we give a reconstruction of the universal quantum invariant using the Heisenberg double. In Section 5 we define colored diagrams and extend the universal quantum invariant to an invariant of colored diagrams up to certain colored moves. Section 6 and Section 7 are devoted to 3-dimensional descriptions of the reconstruction and the extension of the universal quantum invariant. In Section 6 we define colored singular triangulations of topological spaces. The universal quantum invariant can be considered as an invariant of the colored singular triangulations. In Section 7 we define colored ideal triangulations of tangle complements arising from octahedral triangulations, which have been studied in e.g., [CKK14, Yok11] in the context of the hyperbolic geometry.

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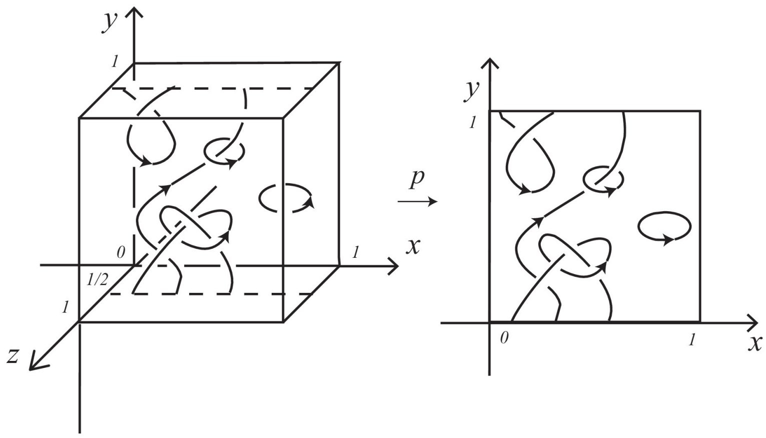

FIGURE 2.1. A tangle and its diagram

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2. Universal quantum invariant

In this paper, a *tangle* means a proper embedding in a cube $[0,1]^3$ of a compact, oriented 1-manifold, whose boundary points are on the two parallel lines $[0,1] \times \{0,1\} \times \{1/2\}$. A *tangle diagram* is a diagram of a tangle obtained from the projection $p\colon (x,y,z) \mapsto (x,y,0)$ to the (x,y)-plane, see Figure 2.1. A *framed* tangle is a tangle equipped with a trivialization of its normal tangent bundle, which is presented in a diagram by the blackboard framing.

2.1. Ribbon Hopf algebras.

Let $(A, \eta_A, m_A, \varepsilon_A, \Delta_A, \gamma_A)$ be a finite dimensional Hopf algebra over a field k, with k-linear maps

$$\begin{aligned} \eta_A &\colon\ k \to A, \\ \varepsilon_A &\colon\ A \to k, \\ m_A &\colon\ A \otimes A \to A, \\ \Delta_A &\colon\ A \to A \otimes A, \\ \gamma_A &\colon\ A \to A, \end{aligned}$$

which are called *unit*, *counit*, *multiplication*, *comultiplication*, and *antipode*, respectively. For simplicity we will omit the subscript A of each map above when there is no confusion.

For distinct integers $1 \le j_1, \dots, j_m \le l$ and $x = \sum x_1 \otimes \cdots \otimes x_m \in A^{\otimes m}$, we use the notation

$$x^{(l)}_{j_1 \dots j_m} = \sum (x_1)_{j_1} \cdots (x_m)_{j_m} \in A^{\otimes l}, \tag{2.1}$$

where $(x_i)_{j_i}$ represents the element in $A^{\otimes l}$ obtained by placing x_i on the j_ith tensorand, i.e.,

$$(x_i)_{j_i} = 1 \otimes \cdots \otimes x_i \otimes \cdots \otimes 1,$$

where x_i is at the j_ith position. For example, for $x = \sum x_1 \otimes x_2 \otimes x_3$, we have $x^{(3)}_{312} = \sum x_2 \otimes x_3 \otimes x_1$. Abusing the notation, we will omit the superscript of $x^{(l)}_{j_1 \dots j_m}$ as $x_{j_1 \dots j_m}$.

For k-modules V, W, we define the symmetry map

$$\tau_{V,W}\colon\ V \otimes W \to W \otimes V, \quad a \otimes b \mapsto b \otimes a. \tag{2.2}$$

A quasi-triangular Hopf algebra $(A, \eta, m, \varepsilon, \Delta, \gamma, R)$ is a Hopf algebra $(A, \eta, m, \varepsilon, \Delta, \gamma)$ with an invertible element $R \in A^{\otimes 2}$, called *the universal R-matrix*, such that

$$\Delta^{\mathrm{op}}(x) = R\Delta(x)R^{-1} \quad \text{for } x \in A,$$
$$(\Delta \otimes 1)(R) = R_{13}R_{23}, \quad (1 \otimes \Delta)(R) = R_{13}R_{12},$$

where $\Delta^{\mathrm{op}} = \tau_{A,A} \circ \Delta$.

A ribbon Hopf algebra $(A, \eta, m, \varepsilon, \Delta, \gamma, R, \mathbf{r})$, see e.g., [Kas95], is a quasi-triangular Hopf algebra $(A, \eta, m, \varepsilon, \Delta, \gamma, R)$ with a central, invertible element $\mathbf{r} \in A$, called *ribbon element*, such that

$$\mathbf{r}^2 = u\gamma(u), \quad \gamma(\mathbf{r}) = \mathbf{r}, \quad \varepsilon(\mathbf{r}) = 1, \quad \Delta(\mathbf{r}) = (R_{21}R)^{-1}(\mathbf{r} \otimes \mathbf{r}),$$

where

$$u = \sum S(\beta)\alpha \tag{2.3}$$

with $R = \sum \alpha \otimes \beta$.

2.2. **Universal quantum invariant for framed tangles.** In this section, we recall the universal quantum invariant [Oht93, Law89, Law90] for framed tangles associated to a ribbon Hopf algebra $(A, \eta, m, \varepsilon, \Delta, \gamma, R, \mathbf{r})$.

Let $T = T_1 \cup \cdots \cup T_n$ be an n-component tangle.

Set

$$N = \mathrm{Span}_k\{ab - ba \mid a, b \in A\} \subset A.$$

For $i = 1, \dots, n$, let

$$A_i = \begin{cases} A & \text{if} \quad \partial T_i \neq \emptyset, \\ A/N & \text{if} \quad \partial T_i = \emptyset. \end{cases}$$

We define the universal quantum invariant $J(T) \in A_1 \otimes \cdots \otimes A_n$ in three steps as follows. We follow the notation in [Suz12].

Step 1. Choose a diagram. We choose a diagram D of T which is obtained by pasting, horizontally and vertically, copies of the fundamental tangles depicted in Figure 2.2.

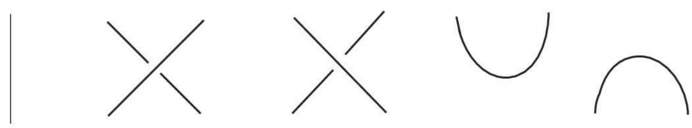


FIGURE 2.2. Fundamental tangles, where the orientation of each strand is arbitrary

Step 2. Attach labels. We attach labels on the copies of the fundamental tangles in the diagram, following the rule described in Figure 2.3, where each γ' should be replaced with γ if the string is oriented upwards, and with the identity otherwise. We do not attach any label to the other copies of fundamental tangles, i.e., to a straight strand and to a local maximum or minimum oriented from right to left.

Step 3. Read the labels. We define the ith tensorand of $J(D)$ as the product of the labels on the ith component of D, where the labels are read off along T_i reversing the orientation, and written from left to right. Here, if T_i is a closed

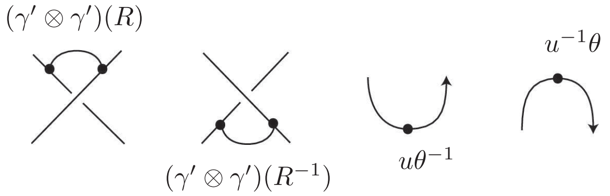


FIGURE 2.3. How to place labels on the fundamental tangles

component, then we choose arbitrary point p_i on T_i and read the label from p_i. The labels on the crossings are read as in Figure 2.4.

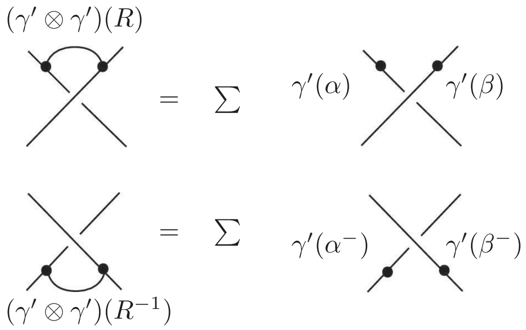


FIGURE 2.4. How to read the labels on crossings, where $R^{-1} = \sum \alpha^- \otimes \beta^-$

As is well known [Oht93], $J(T) := J(D)$ does not depend on the choice of the diagram and the base points p_i, and thus defines an isotopy invariant of tangles.

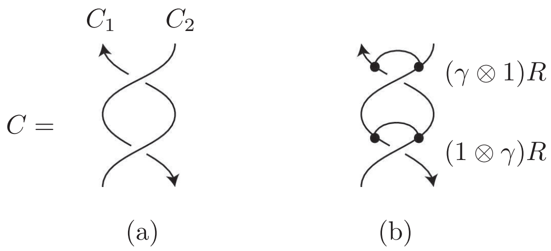


FIGURE 2.5. (a) A tangle diagram C, (b) The label put on C

For example, for the tangle $C = C_1 \cup C_2$ shown in Figure 2.5, we have

$$J(C) = \sum \gamma(\alpha)\gamma(\beta') \otimes \alpha'\beta, \tag{2.4}$$

where $R = \sum \alpha \otimes \beta = \sum \alpha' \otimes \beta'$.

3. DRINFELD DOUBLE AND HEISENBERG DOUBLE

Let $(A, \eta, m, \varepsilon, \Delta, \gamma)$ be a finite dimensional Hopf algebra. Let $A^* = \mathrm{Hom}_k(A, k)$. Define the pairing

$$\langle\ ,\ \rangle\colon\ A^* \otimes A \to k, \quad f \otimes x \mapsto f(x), \tag{3.1}$$

and extend it to

$$\langle\ ,\ \rangle\colon\ (A^*)^{\otimes n} \otimes A^{\otimes n} \to k,$$

for $n \geq 1$, by

$$\langle f_1 \otimes \cdots \otimes f_n, x_1 \otimes \cdots \otimes x_n \rangle = \langle f_1, x_1 \rangle \cdots \langle f_n, x_n \rangle.$$

Then the dual Hopf algebra

$$A^* = (A^*, \eta_{A^*} = \varepsilon^*, m_{A^*} = \Delta^*, \varepsilon_{A^*} = \eta^*, \Delta_{A^*} = m^*, \gamma_{A^*} = \gamma^*)$$

is defined using the transposes of the morphisms of A, i.e., is defined uniquely by

$$\begin{aligned} \langle \varepsilon^*(a), x \rangle &= a\varepsilon(x), \quad a \in k, x \in A, \\ \langle \Delta^*(f \otimes g), x \rangle &= \langle f \otimes g, \Delta(x) \rangle, \quad f, g \in A^*, x \in A, \\ \eta^*(f)a &= \langle f, \eta(a) \rangle, \quad f \in A^*, a \in k, \\ \langle m^*(f), x \otimes y \rangle &= \langle f, m(x \otimes y) \rangle, \quad f \in A^*, x, y \in A, \\ \langle \gamma^*(f), x \rangle &= \langle f, \gamma(x) \rangle, \quad f \in A^*, x \in A. \end{aligned}$$

3.1. **Drinfeld double and Yang-Baxter equation.** For any finite dimensional Hopf algebra with invertible antipode, the Drinfeld quantum double construction gives a quasi-triangular Hopf algebra [Dri87]. Here, we follow the notation in [Kass95].

Let $(A, \eta, m, \varepsilon, \Delta, \gamma, \gamma^{-1})$ be a finite dimensional Hopf algebra with invertible antipode, $A^{\mathrm{op}} = (A, \eta, m^{\mathrm{op}}, \varepsilon, \Delta, \gamma^{-1}, \gamma)$ the opposite Hopf algebra and $(A^{\mathrm{op}})^* = (A^*, \varepsilon^*, \Delta^*, \eta^*, (m^{\mathrm{op}})^*, (\gamma^{-1})^*, \gamma^*)$ the dual of the opposite Hopf algebra, where $m^{\mathrm{op}} = m \circ \tau_{A,A}$. For simplicity, we set

$$\bar{\gamma} = \gamma^{-1}.$$

Let $\Delta^{(0)} = \mathrm{id}$ and $\Delta^{(n)} = (\Delta \otimes 1^{\otimes n-1})\Delta^{(n-1)}$ for $n \geq 1$. In what follows, for $x \in A$ or $x \in A^*$, we use the notation

$$\begin{aligned} \Delta(x) = \Delta^{(1)}(x) &= \sum x' \otimes x'' = \sum x^{(1)} \otimes x^{(2)}, \\ (\Delta \otimes 1)\Delta(x) = \Delta^{(2)}(x) &= \sum x' \otimes x'' \otimes x''' = \sum x^{(1)} \otimes x^{(2)} \otimes x^{(3)}, \\ \Delta^{(n)} &= \sum x^{(1)} \otimes \cdots \otimes x^{(n+1)}, \end{aligned}$$

for $n \geq 3$. We have

$$(m^{\mathrm{op}})^*(f) = \Delta^{\mathrm{op}}(f) = \sum f'' \otimes f'$$

for $f \in (A^{\mathrm{op}})^*$. [2]

There is a unique left action

$$A \otimes (A^{\mathrm{op}})^* \to (A^{\mathrm{op}})^*, \quad a \otimes f \mapsto a \cdot f,$$

such that

$$\langle a \cdot f, x \rangle = \sum \langle f, \bar{\gamma}(a'') x a' \rangle,$$

for $a, x \in A$ and $f \in (A^{\mathrm{op}})^*$, which induces the left A-module coalgebra structure on $(A^{\mathrm{op}})^*$. Also, there is a unique right action

$$A \otimes (A^{\mathrm{op}})^* \to A, \quad a \otimes f \mapsto a^f,$$

such that

$$a^f = \sum f(\bar{\gamma}(a''')a')a''$$

for $a \in A$ and $f \in (A^{\mathrm{op}})^*$, which induces the right $(A^{\mathrm{op}})^*$-module coalgebra structure on A.

The Drinfeld double

$$D(A) = ((A^{\mathrm{op}})^* \otimes A, \eta_{D(A)}, m_{D(A)}, \varepsilon_{D(A)}, \Delta_{D(A)}, \gamma_{D(A)}, R)$$

[2] In [Kass95], he uses the notation $\Delta^{\mathrm{op}}(f) = \sum f' \otimes f''$.

is the quasi-triangular Hopf algebra defined as the bicrossed product of A and $(A^{\mathrm{op}})^*$. Its unit, counit, and comultiplication are given by these of $(A^{\mathrm{op}})^* \otimes A$, i.e., we have

$$\begin{aligned}\eta_{D(A)}(1) &= \eta_{(A^{\mathrm{op}})^*\otimes A}(1) = 1 \otimes 1,\\ \varepsilon_{D(A)}(f \otimes a) &= \varepsilon_{(A^{\mathrm{op}})^*\otimes A}(f \otimes a) = f(1)\varepsilon(a),\\ \Delta_{D(A)}(f \otimes a) &= \Delta_{(A^{\mathrm{op}})^*\otimes A}(f \otimes a) = \sum f'' \otimes a' \otimes f' \otimes a'',\end{aligned}$$

for $a \in A$ and $f \in (A^{\mathrm{op}})^*$. Its multiplication is given by

$$m_{D(A)}\left((f \otimes a) \otimes (g \otimes b)\right) = \sum f(a' \cdot g'') \otimes a''^{g'} b \tag{3.2}$$

$$= \sum fg(\bar{\gamma}(a''')?a') \otimes a'' b, \tag{3.3}$$

for $a, b \in A$ and $f, g \in (A^{\mathrm{op}})^*$, where the question mark ? denotes a place of the variable. Its antipode is given by

$$\gamma_{D(A)}(f \otimes a) = \sum \gamma(a'') \cdot \bar{\gamma}^*(f') \otimes \gamma(a')^{\bar{\gamma}^*(f'')},$$

for $a \in A$ and $f \in (A^{\mathrm{op}})^*$.

Fix a basis $\{e_a\}_{a\in\mathcal{I}}$ of A and its dual basis $\{e^a\}_{a\in\mathcal{I}}$ of A^*. The universal R-matrix is defined as the canonical element

$$R = \sum_a (1 \otimes e_a) \otimes (e^a \otimes 1) \in D(A) \otimes D(A).$$

3.2. **Heisenberg double and pentagon relation.** Let A be a finite dimensional Hopf algebra with an invertible antipode as in the previous section. The Heisenberg double

$$H(A) = (A^* \otimes A, \eta_{H(A)}, m_{H(A)})$$

is the algebra with the unit $\eta_{H(A)}(1) = \eta_{A^*\otimes A}(1) = 1 \otimes 1$ and the multiplication

$$m_{H(A)}\left((f \otimes a) \otimes (g \otimes b)\right) = \sum fg(?a') \otimes a'' b, \tag{3.4}$$

for $a, b \in A$ and $f, g \in (A^{\mathrm{op}})^*$.

Kashaev showed the following.

Theorem 3.1 (Kashaev [Kas97]). *The canonical element*

$$S = \sum_a (1 \otimes e_a) \otimes (e^a \otimes 1) \in H(A) \otimes H(A)$$

satisfies the pentagon relation

$$S_{12}S_{13}S_{23} = S_{23}S_{12} \quad \in H(A)^{\otimes 3}. \tag{3.5}$$

3.3. **Drinfeld double and Heisenberg double.** Let

$$H(A^*) = (A \otimes A^*, \eta_{H(A^*)}, m_{H(A^*)})$$

be the Heisenberg double of the dual Hopf algebra A^* of A, where we identify $(A^*)^*$ and A in the standard way.

Set $A^{\mathrm{opcop}} = (A, \eta, m^{\mathrm{op}}, \varepsilon, \Delta^{\mathrm{op}}, \gamma, \gamma^{-1})$. We have the following lemma.

Lemma 3.2. *The algebras $H(A^*)$ and $H(A)^{\mathrm{op}}$ are isomorphic via the unique isomorphism $\Gamma \circ \tau$ such that*

$$\begin{aligned}\tau = \tau_{A^*,A}\colon\ H(A^*) \to H(A^{\mathrm{opcop}})^{\mathrm{op}}, &\quad x \otimes f \mapsto f \otimes x,\\ \Gamma = \bar{\gamma}^* \otimes \gamma\colon\ H(A^{\mathrm{opcop}})^{\mathrm{op}} \to H(A)^{\mathrm{op}}, &\quad f \otimes x \mapsto \bar{\gamma}^*(f) \otimes \gamma(x).\end{aligned}$$

Proof. We have

$$
\begin{aligned}
\tau(x\otimes f)\cdot_{H(A^{\mathrm{opcop}})^{\mathrm{op}}}\tau(y\otimes g) &= (f\otimes x)\cdot_{H(A^{\mathrm{opcop}})^{\mathrm{op}}}(g\otimes y)\\
&= (g\otimes y)\cdot_{H(A^{\mathrm{opcop}})}(f\otimes x)\\
&= \sum g\cdot_{(A^*)^{\mathrm{op}}}\langle f, ?\cdot_{A^{\mathrm{op}}} y''\rangle\otimes y'\cdot_{A^{\mathrm{op}}} x\\
&= \sum\langle f, y''?\rangle g\otimes xy'\\
&= \sum\langle f', y''\rangle f''g\otimes xy'\\
&= \sum f''g\otimes xy'\langle f', y''\rangle\\
&= \sum\tau\left(xy'\langle f', y''\rangle\otimes f''g\right)\\
&= \sum\tau\left(x\langle ?f', y\rangle\otimes f''g\right)\\
&= \tau\left((x\otimes f)\cdot_{H(A^*)}(y\otimes g)\right),
\end{aligned}
$$

and we have

$$
\begin{aligned}
\Gamma(f\otimes x)\cdot_{H(A)^{\mathrm{op}}}\Gamma(g\otimes y) &= (\bar{\gamma}^*(f)\otimes\gamma(x))\cdot_{H(A)^{\mathrm{op}}}(\bar{\gamma}^*(g)\otimes\gamma(y))\\
&= (\bar{\gamma}^*(g)\otimes\gamma(y))\cdot_{H(A)}(\bar{\gamma}^*(f)\otimes\gamma(x))\\
&= \sum\bar{\gamma}^*(g)\langle\bar{\gamma}^*(f), ?\gamma(y)'\rangle\otimes\gamma(y)''\gamma(x)\\
&= \sum\bar{\gamma}^*(g)\bar{\gamma}^*(f)'\langle\bar{\gamma}^*(f)'', \gamma(y)'\rangle\otimes\gamma(y)''\gamma(x)\\
&= \sum\bar{\gamma}^*(g)\bar{\gamma}^*(f'')\langle\bar{\gamma}^*(f'), \gamma(y'')\rangle\otimes\gamma(y')\gamma(x)\\
&= \sum(\bar{\gamma}^*\otimes\gamma)\left(\langle f', y''\rangle f''g\otimes xy'\right)\\
&= \sum(\bar{\gamma}^*\otimes\gamma)\left(\langle f, y''?\rangle g\otimes xy'\right)\\
&= \sum\Gamma\left(g\cdot_{(A^*)^{\mathrm{op}}}\langle f, ?\cdot_{A^{\mathrm{op}}} y''\rangle\otimes y'\cdot_{A^{\mathrm{op}}} x\right)\\
&= \Gamma\left((g\otimes y)\cdot_{H(A^{\mathrm{opcop}})}(f\otimes x)\right)\\
&= \Gamma\left((f\otimes x)\cdot_{H(A^{\mathrm{opcop}})^{\mathrm{op}}}(g\otimes y)\right),
\end{aligned}
$$

which completes the proof. □

Set

$$\phi(1\otimes e_a)=\sum 1\otimes e_a'\otimes 1\otimes\gamma(e_a'')\in H(A)\otimes H(A)^{\mathrm{op}}, \tag{3.6}$$

$$\phi(e^a\otimes 1)=\sum (e^a)''\otimes 1\otimes\bar{\gamma}^*((e^a)')\otimes 1\in H(A)\otimes H(A)^{\mathrm{op}}. \tag{3.7}$$

Kashaev [Kas97] showed that the Drinfeld double $D(A)$ can be realized as a subalgebra in the tensor product $H(A)\otimes H(A)^{\mathrm{op}}$ of the Heisenberg double $H(A)$ and its opposite algebra $H(A)^{\mathrm{op}}$ as follows. [3]

Theorem 3.3 (Kashaev [Kas97])**.** *There is a unique algebra homomorphism*

$$\phi\colon\ D(A)\to H(A)\otimes H(A)^{\mathrm{op}} \tag{3.8}$$

extending (3.6) and (3.7).

Proposition 3.4. *We have*

(3.9)

$$\phi = m_{H(A)\otimes H(A)^{\mathrm{op}}}\circ\left((1\otimes\eta)^{\otimes 2}\otimes(\eta\otimes 1)^{\otimes 2}\right)\circ(1\otimes\bar{\gamma}^*\otimes 1\otimes\gamma)\circ(\Delta^{\mathrm{op}}\otimes\Delta),$$

[3] In [Kas97] he uses $H(A^*)$ instead of $H(A)^{\mathrm{op}}$.

i.e., we have

$$\begin{aligned}\phi(f\otimes x) &= \sum \langle \bar{\gamma}^*\left(f''\right)', \gamma(x'')'\rangle f''' \otimes x' \otimes \bar{\gamma}^*\left(f''\right)' \otimes \gamma(x'')'' \\ &= \sum \langle f', x'''\rangle f''' \otimes x' \otimes \bar{\gamma}^*(f'') \otimes \gamma(x''),\end{aligned}$$

for $f \in A^*$ and $x \in A$.

Proof. Let ϕ' denote the right hand side of (3.9). We have only to prove that (a) $\phi = \phi'$ on generators of $D(A)$, and (b) ϕ' is an algebra homomorphism.

For (a), we have

$$\begin{aligned}\phi'(1\otimes e_a) &= \sum \langle 1, e_a'''\rangle 1 \otimes e_a' \otimes 1 \otimes \gamma\left(e_a''\right)). \\ &= \sum 1 \otimes e_a' \otimes 1 \otimes \gamma(e_a'') \\ &= \phi(1\otimes e_a), \\ \phi'(e^a \otimes 1) &= \sum \langle (e^a)', 1\rangle (e^a)''' \otimes 1 \otimes (\bar{\gamma}^*(e^a)'') \otimes 1. \\ &= \sum (e^a)'' \otimes 1 \otimes \bar{\gamma}^*((e^a)') \otimes 1 \\ &= \phi(e^a \otimes 1).\end{aligned}$$

For (b), we have

$$\begin{aligned}\phi'(1\otimes x)\phi'(1\otimes y) &= \left(\sum 1 \otimes x' \otimes 1 \otimes \gamma(x'')\right) \cdot_{H(A)\otimes H(A)^{\mathrm{op}}} \left(\sum 1 \otimes y' \otimes 1 \otimes \gamma(y'')\right) \\ &= \sum 1 \otimes x'y' \otimes 1 \otimes \gamma(x'') \cdot_{A^{\mathrm{op}}} \gamma(y'') \\ &= \sum 1 \otimes (xy)' \otimes 1 \otimes \gamma((xy)'') \\ &= \phi'\left((1\otimes x)\cdot_{D(A)} (1\otimes y)\right),\end{aligned}$$

$$\begin{aligned}\phi'(f\otimes 1)\phi'(g\otimes 1) &= \left(\sum f'' \otimes 1 \otimes \bar{\gamma}^*(f') \otimes 1\right) \cdot_{H(A)\otimes H(A)^{\mathrm{op}}} \left(\sum g'' \otimes 1 \otimes \bar{\gamma}^*(g') \otimes 1\right) \\ &= \sum f''g'' \otimes 1 \otimes \bar{\gamma}^*(f') \cdot_{(A^*)^{\mathrm{op}}} \bar{\gamma}^*(g') \otimes 1 \\ &= \sum (fg)'' \otimes 1 \otimes \bar{\gamma}^*((fg)') \otimes 1 \\ &= \phi'\left((f\otimes 1)\cdot_{D(A)} (g\otimes 1)\right),\end{aligned}$$

$$\begin{aligned}\phi'(f\otimes 1)\phi'(1\otimes x) &= \left(\sum f'' \otimes 1 \otimes \bar{\gamma}^*(f') \otimes 1\right) \cdot_{H(A)\otimes H(A)^{\mathrm{op}}} \left(\sum 1 \otimes x' \otimes 1 \otimes \gamma(x'')\right) \\ &= \sum f'' \otimes x' \otimes \langle \bar{\gamma}^*(f')'', \gamma(x'')'\rangle \bar{\gamma}^*\left(f'\right)' \otimes \gamma(x'')'' \\ &= \sum \langle f', x'''\rangle f''' \otimes x' \otimes \bar{\gamma}^*\left(f''\right) \otimes \gamma(x'') \\ &= \phi'(f\otimes x) \\ &= \phi'\left((f\otimes 1)\cdot_{D(A)} (1\otimes x)\right),\end{aligned}$$

and

$$\begin{aligned}
\phi'(1\otimes x)\phi'(f\otimes 1) &= \left(\sum 1\otimes x^{(1)}\otimes 1\otimes\gamma(x^{(2)})\right)\cdot_{H(A)\otimes H(A)^{\mathrm{op}}}\left(\sum f^{(2)}\otimes 1\otimes\bar{\gamma}^*(f^{(1)})\otimes 1\right)\\
&= \sum\langle f^{(3)},x^{(1)}\rangle f^{(2)}\otimes x^{(2)}\otimes\bar{\gamma}^*\left(f^{(1)}\right)\otimes\gamma(x^{(3)})\\
&= \sum\langle f^{(4)},x^{(1)}\rangle\varepsilon(f^{(1)})\varepsilon\left(x^{(4)}\right)f^{(3)}\otimes x^{(2)}\otimes\bar{\gamma}^*\left(f^{(2)}\right)\otimes\gamma(x^{(3)})\\
&= \sum\langle f^{(4)},x^{(1)}\rangle\langle f^{(1)},\bar{\gamma}(x^{(5)})x^{(4)}\rangle f^{(3)}\otimes x^{(2)}\otimes\bar{\gamma}^*\left(f^{(2)}\right)\otimes\gamma(x^{(3)})\\
&= \sum\langle f^{(5)},x^{(1)}\rangle\langle f^{(1)},\bar{\gamma}(x^{(5)})\rangle\langle f^{(2)},x^{(4)}\rangle f^{(4)}\otimes x^{(2)}\otimes\bar{\gamma}^*\left(f^{(3)}\right)\otimes\gamma(x^{(3)})\\
&= \phi'\left(\langle f^{(1)},\gamma(x^{(3)})\rangle\langle f^{(3)},x^{(1)}\rangle f^{(2)}\otimes x^{(2)}\right),\\
&= \phi'\left((1\otimes x)\cdot_{D(A)}(f\otimes 1)\right),
\end{aligned}$$

where the fourth identity follows from $m^{\mathrm{op}}(1\otimes\bar{\gamma})\Delta=\eta\varepsilon$.

Thus we have the assertion. □

Set

$$\begin{aligned}
\hat{R} = \phi^{\otimes 2}(R) &= \sum_a \phi(1\otimes e_a)\otimes\phi(e^a\otimes 1)\\
&= \sum 1\otimes e_a'\otimes 1\otimes\gamma(e_a'')\otimes(e^a)''\otimes 1\otimes\bar{\gamma}^*((e^a)')\otimes 1 \quad \in (H(A)\otimes H(A)^{\mathrm{op}})^{\otimes 2}.
\end{aligned}$$

Since $\phi^{\otimes 2}$ is an algebra homomorphism, the element $\hat{R}$ also satisfies the quantum Yang-Baxter equation:

$$\hat{R}_{12}\hat{R}_{13}\hat{R}_{23}=\hat{R}_{23}\hat{R}_{13}\hat{R}_{12}, \tag{3.10}$$

where we use the notation (2.1) treating $H(A)\otimes H(A)^{\mathrm{op}}$ as one algebra. If we treat $H(A)\otimes H(A)^{\mathrm{op}}$ as the tensor of $H(A)$ and $H(A)^{\mathrm{op}}$, we have

$$\hat{R}_{1234}\hat{R}_{1256}\hat{R}_{3456}=\hat{R}_{3456}\hat{R}_{1256}\hat{R}_{1234}.$$

Set

$$\tilde{e}_a := \gamma(e_a),\quad \tilde{e}^b := \bar{\gamma}^*(e^b),$$

and set

$$\begin{aligned}
S' &= \sum(1\otimes\tilde{e}_a)\otimes(e^a\otimes 1) \quad \in H(A)^{\mathrm{op}}\otimes H(A),\\
S'' &= \sum(1\otimes e_a)\otimes(\tilde{e}^a\otimes 1) \quad \in H(A)\otimes H(A)^{\mathrm{op}},\\
\tilde{S} &= \sum(1\otimes\tilde{e}_a)\otimes(\tilde{e}^a\otimes 1) \quad \in H(A)^{\mathrm{op}}\otimes H(A)^{\mathrm{op}}.
\end{aligned}$$

Kashaev showed the following.

Theorem 3.5 (Kashaev [Kas97])**.** *We have*

$$\hat{R}=S''_{14}S_{13}\tilde{S}_{24}S'_{23} \quad \in (H(A)\otimes H(A)^{\mathrm{op}})^{\otimes 2}. \tag{3.11}$$

Proposition 3.6 (Kashaev [Kas97])**.** *The quantum Yang-Baxter equation (3.10) in $(H(A)\otimes H(A)^{\mathrm{op}})^{\otimes 3}$ is a consequence of the following variations of the pentagon equation for the tensors S, S', S'' and $\tilde{S}$:*

$$S_{23}S_{12}=S_{12}S_{13}S_{23},\quad S_{23}S'_{12}=S'_{12}S'_{13}S_{23}, \tag{3.12}$$

$$S''_{23}S_{12}=S_{12}S''_{13}S''_{23},\quad S''_{23}S'_{12}=S'_{12}S''_{13}S''_{23}, \tag{3.13}$$

and

$$S'_{23}S_{13}S''_{12}=S''_{12}S'_{23},\quad S'_{23}S'_{13}\tilde{S}_{12}=\tilde{S}_{12}S'_{23}, \tag{3.14}$$

$$\tilde{S}_{23}S''_{13}S''_{12}=S''_{12}S''_{23},\quad \tilde{S}_{23}\tilde{S}_{13}\tilde{S}_{12}=\tilde{S}_{12}\tilde{S}_{23}. \tag{3.15}$$

4. Reconstruction of the universal quantum invariant

Let $D(A)$ be the Drinfeld double of A. Recall from (2.3) the element $u = \sum \gamma(\beta)\alpha = \sum \bar{\gamma}^*(e^a) \otimes e_a$ with $R = \sum \alpha \otimes \beta = \sum (1 \otimes e_a) \otimes (e^a \otimes 1)$. We have a ribbon Hopf algebra

$$D(A)^\theta = D(A)[\theta]/\left(\theta^2 - u\gamma(u)\right)$$

with the ribbon element θ (e.g., [Kass95]).

We also consider the algebra

$$\left(H(A) \otimes H(A)^{\mathrm{op}}\right)^{\bar{\theta}} = \left(H(A) \otimes H(A)^{\mathrm{op}}\right)[\bar{\theta}]/\left(\bar{\theta}^2 - \phi(u\gamma(u))\right),$$

and extend ϕ to the map

$$\phi \colon D(A)^\theta \to \left(H(A) \otimes H(A)^{\mathrm{op}}\right)^{\bar{\theta}}$$

by $\phi(\theta) = \bar{\theta}$.

In this section, we define (non-framed) tangle invariant J' using $\left(H(A) \otimes H(A)^{\mathrm{op}}\right)^{\bar{\theta}}$, which turns out to be the image of tensor power of ϕ of a non-framed version of the universal invariant associated to $D(A)^\theta$ (Theorem 4.1).

In what follows, for simplicity, we use the notation

$$fx = f \otimes x \in A^* \otimes A,$$

for $f \in A^*$ and $x \in A$. In particular we have

$$S = \sum_a e^a \otimes e_a, \quad S' = \sum_a \tilde{e}^a \otimes e_a, \quad S'' = \sum_a e^a \otimes \tilde{e}_a, \quad \tilde{S} = \sum_a \tilde{e}^a \otimes \tilde{e}_a.$$

4.1. Reconstruction of the universal quantum invariant using the Heisenberg double.

Let $T = T_1 \cup \cdots \cup T_n$ be an n-component tangle. Similarly to Section 2.2, set

$$N_{(H \otimes H^{\mathrm{op}})^{\bar{\theta}}} = \mathrm{Span}_k\{ab - ba \mid a, b \in \left(H(A) \otimes H(A)^{\mathrm{op}}\right)^{\bar{\theta}}\} \subset \left(H(A) \otimes H(A)^{\mathrm{op}}\right)^{\bar{\theta}}.$$

For $i = 1, \ldots, n$, let

$$\left(H(A) \otimes H(A)^{\mathrm{op}}\right)_i^{\bar{\theta}} = \begin{cases} H(A) \otimes H(A)^{\mathrm{op}} & \text{if } \partial T_i \neq \emptyset, \\ \left(H(A) \otimes H(A)^{\mathrm{op}}\right)^{\bar{\theta}} / N_{(H(A) \otimes H(A)^{\mathrm{op}})^{\bar{\theta}}} & \text{if } \partial T_i = \emptyset. \end{cases}$$

Take a diagram D of T. We define an element $J''(D) \in \bigotimes_i \left(H(A) \otimes H(A)^{\mathrm{op}}\right)_i^{\bar{\theta}}$ modifying the definition of $J(T)$ as follows.

We duplicate D and thicken the left strands following the orientation, and denote the result by $\zeta(D)$. For examples, recall the tangle C in Figure 4.1 (a), and let T_{41} be the $(1,1)$-tangle in Figure 4.2 (a) whose closure is the figure eight knot. See Figure 4.1 (b) and 4.2 (b) for $\zeta(C)$ and $\zeta(T_{41})$, respectively.

Then we put labels on crossings as in Figure 4.3, where each γ' and each $(\bar{\gamma}^*)'$ should be replaced with γ and $\bar{\gamma}^*$, respectively, if the string is oriented upwards, and with the identities otherwise.

We define the $(2i-1)$st and the $2i$th tensorands of $J''(D)$ as the product of the labels on the thin and the thick strands, respectively, obtained by duplicating T_i, where the labels are read off reversing the orientation, and written from left to right. Here, if T_i is a closed component, then we choose a point p on T_i and denote by p' (resp. p'') the image of p by the duplicating procedure on the thin (resp. thick) strand. We read the labels of the thin (resp. thick) strand from p' (resp. p'').

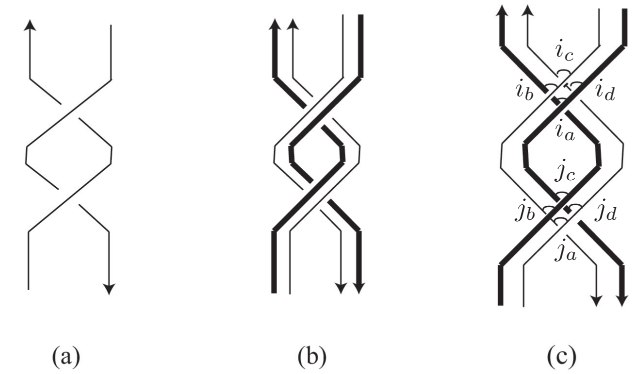


FIGURE 4.1. (a) A tangle C, (b) The diagram $\zeta(C)$, (c) Parameters for $\zeta(C)$

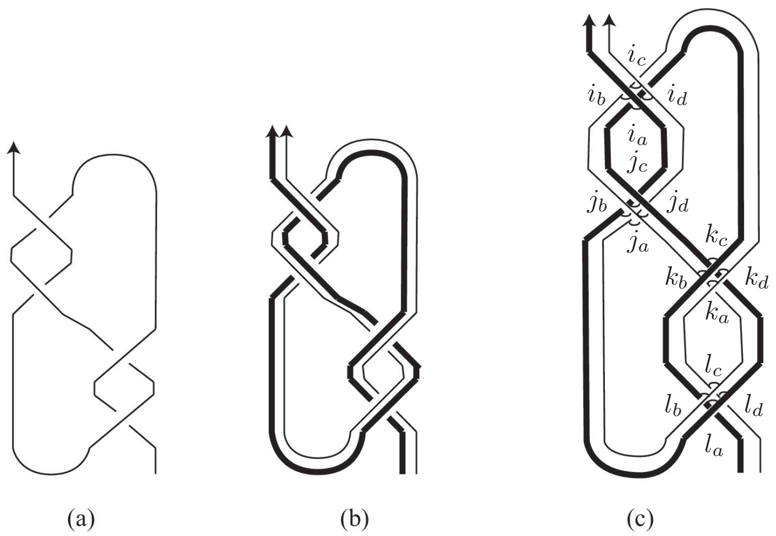


FIGURE 4.2. (a) A tangle T_{41}, (b) The diagram $\zeta(T_{41})$, (c) Parameters for $\zeta(T_{41})$

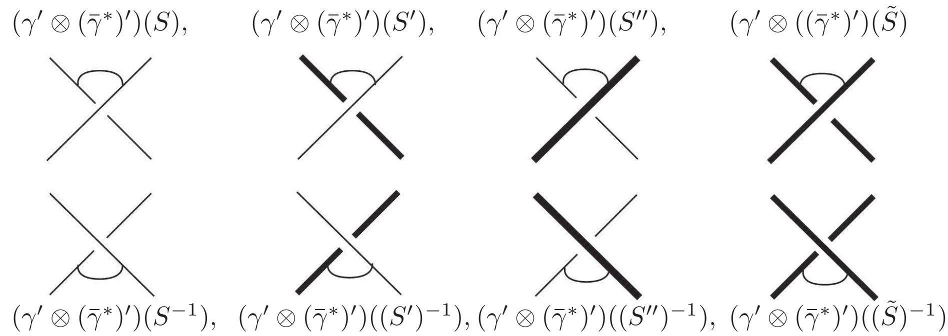


FIGURE 4.3. Labels on crossings

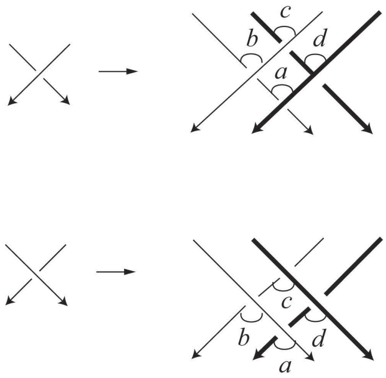


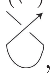
FIGURE 4.4. Labels on the colored diagrams $\zeta(c^{\pm})$ associated to positive and negative crossings $c^{\pm}$

Let $f(D_i) = \#\{\text{positive self crossings of } D_i\} - \#\{\text{negative self crossings of } D_i\}$ be the framing of D_i. Set

$$J'(D) = \left(\prod_i \bar{\theta}_i^{f(D_i)} \right) J''(D) \in \bigotimes_i (H(A) \otimes H(A)^{\mathrm{op}})_i^{\bar{\theta}}.$$

Let

$$(\leftarrow)\colon \{\text{tangle diagrams}\} \to \{\text{tangle diagrams}\}, \quad D \mapsto D_{(\leftarrow)}, \tag{4.1}$$

where $D_{(\leftarrow)}$ is the diagram obtained from D by replacing each of  and  with  and , respectively.

Theorem 4.1. *Let T be an n-component framed tangle, and let T^0 denote T with 0-framing. Let D be a diagram of T. We have*

$$J'(D_{(\leftarrow)}) = \phi^{\otimes n} \circ J(T^0) \quad \in \bigotimes_i (H(A) \otimes H(A)^{\mathrm{op}})_i^{\bar{\theta}}.$$

Proof. We have only to prove the compatibility of the label attached to each crossing of $D_{(\leftarrow)}$, since we used a tangle diagram without  and , and the processes to read the labels are compatible.

Consider a positive or a negative crossing c between two distinct strands oriented downwards. We have $J'(c) = J''(c) = \phi^{\otimes 2} \circ J(c)$ since

$$\hat{R}_{1234} = S''_{14} S_{13} \tilde{S}_{24} S'_{23} = \sum_{a,b,c,d} e_a e_b \otimes \tilde{e}_d \tilde{e}_c \otimes e^b e^c \otimes \tilde{e}^a \tilde{e}^d,$$

$$\hat{R}^{-1}_{1234} = (S'_{23})^{-1} (\tilde{S}_{24})^{-1} (S_{13})^{-1} (S''_{14})^{-1} = \sum_{a,b,c,d} u_b u_c \otimes \tilde{u}_a \tilde{u}_d \otimes u^a u^b \otimes \tilde{u}^d \tilde{u}^c,$$

where u_a, u^a, $\tilde{u}_a$, $\tilde{u}^a$ are defined by

$$\sum_a u_a \otimes u^a = S^{-1} = \sum_a \gamma(e_a) \otimes e^a, \quad \sum_a \tilde{u}_a \otimes u^a = (S')^{-1} = \sum_a \gamma(\tilde{e}_a) \otimes e^a,$$

$$\sum_a u_a \otimes \tilde{u}^a = (S'')^{-1} = \sum_a \gamma(e_a) \otimes \tilde{e}^a, \quad \sum_a \tilde{u}_a \otimes \tilde{u}^a = \tilde{S}^{-1} = \sum_a \gamma(\tilde{e}_a) \otimes \tilde{e}^a,$$

see Figure 4.4.

Consider a positive (resp. negative) self crossing c with strands oriented downwards. We have $J'(c) = \bar{\theta} J''(c) = \phi^{\otimes 2} \circ J(c^0)$ since in the definition of J, cancelling

FIGURE 5.1. A symmetry, where the orientation of each strand is arbitrary

the framing of c by adding a negative (resp. positive) kink corresponds to multiplying θ (resp. θ^{-1}), and in the definition of J' the last framing adjustment calls the multiplication of $\bar{\theta}$ (resp. $\bar{\theta}^{-1}$).

For a crossing c with other orientations, $J'(c) = \phi^{\otimes 2} \circ J(c^0)$ follows similarly from

$$\begin{aligned}
\phi^{\otimes 2} \circ (\gamma_{D(A)} \otimes 1)(R) &= \sum_{a,b,c,d} \gamma(\tilde{e}_c)\gamma(\tilde{e}_d) \otimes \gamma(e_b)\gamma(e_a) \otimes e^b e^c \otimes \tilde{e}^a \tilde{e}^d, \\
\phi^{\otimes 2} \circ (1 \otimes \gamma_{D(A)})(R) &= \sum_{a,b,c,d} e_a e_b \otimes \tilde{e}_d \tilde{e}_c \otimes \bar{\gamma}^*(\tilde{e}^d)\bar{\gamma}^*(\tilde{e}^a) \otimes \bar{\gamma}^*(e^c)\bar{\gamma}^*(e^b), \\
\phi^{\otimes 2} \circ (\gamma_{D(A)} \otimes 1)(R^{-1}) &= \sum_{a,b,c,d} \gamma(\tilde{u}_d)\gamma(\tilde{u}_a) \otimes \gamma(u_c)\gamma(u_b) \otimes u^a u^b \otimes \tilde{u}^d \tilde{u}^c, \\
\phi^{\otimes 2} \circ (1 \otimes \gamma_{D(A)})(R^{-1}) &= \sum_{a,b,c,d} u_b u_d \otimes \tilde{u}_a \tilde{u}_d \otimes \bar{\gamma}^*(\tilde{u}^c)\bar{\gamma}^*(\tilde{u}^d) \otimes \bar{\gamma}^*(u^b)\bar{\gamma}^*(u^a),
\end{aligned}$$

which completes the proof. $\square$

For the example with C, with the parameters as in Figure 4.1 (c), we have

$$\begin{aligned}
J'_C = J''_C = \sum_{i_a,i_b,i_c,i_d,j_a,j_b,j_c,j_d} &\gamma(e_{i_c})\gamma(e_{i_d})\gamma^*(e^{j_d})\gamma^*(e^{j_a}) \otimes \gamma(\tilde{e}_{i_b})\gamma(\tilde{e}_{i_a})\gamma^*(\tilde{e}^{j_c})\gamma^*(\tilde{e}^{j_b}) \\
&\otimes e_{j_a} e_{j_b} e^{i_b} e^{i_c} \otimes \tilde{e}_{j_d} \tilde{e}_{j_c} \tilde{e}^{i_a} \tilde{e}^{i_d}
\end{aligned}$$

For the example with T_{41}, with the parameters as in Figure 4.2 (c), we have

$$\begin{aligned}
J'_{T_{41}} = J''_{T_{41}} = &\sum_{i_a,i_b,i_c,i_d,j_a,j_b,j_c,j_d,k_a,k_b,k_c,k_d,l_a,l_b,l_c,l_d} \bar{\gamma}^*(u^{i_c})\bar{\gamma}^*(u^{i_d})\gamma(u_{j_d})\gamma(u_{j_a}) e^{l_b} e^{l_c} e_{k_a} e_{k_b} \\
&\times u^{j_a} u^{j_b} u_{i_b} u_{i_c} \bar{\gamma}^*(e^{k_d})\bar{\gamma}^*(e^{k_a})\gamma(e_{l_c})\gamma(e_{l_d}) \\
&\otimes \bar{\gamma}^*(\tilde{u}^{i_b})\bar{\gamma}^*(\tilde{u}^{i_a})\gamma(\tilde{u}_{j_c})\gamma(\tilde{u}_{j_b}) \tilde{e}^{l_a} \tilde{e}^{l_d} \tilde{e}_{k_d} \tilde{e}_{k_c} \\
&\times \tilde{u}^{j_d} \tilde{u}^{j_c} \tilde{u}_{i_a} \tilde{u}_{i_d} \bar{\gamma}^*(\tilde{e}^{k_c})\bar{\gamma}^*(\tilde{e}^{k_b})\gamma(\tilde{e}_{l_b})\gamma(\tilde{e}_{l_a}).
\end{aligned}$$

5. EXTENSION OF THE UNIVERSAL QUANTUM INVARIANT TO AN INVARIANT FOR COLORED DIAGRAMS

In this section we define *colored diagrams* and extend the map J'' to an invariant for colored diagrams up to *colored moves*.

5.1. Colored diagrams and an extension of J''.

In what follows, we consider also a virtual crossing as in Figure 5.1, which we call a *symmetry*. By a *crossing* we mean only a real crossing.

A *colored diagram* Z is a virtual tangle diagram consisting of *thin* strands and *thick* strands, which is obtained by pasting, horizontally and vertically, copies of fundamental tangle diagrams in Figure 2.2 and copies of the symmetry, where the thickness of each strand are arbitrary.

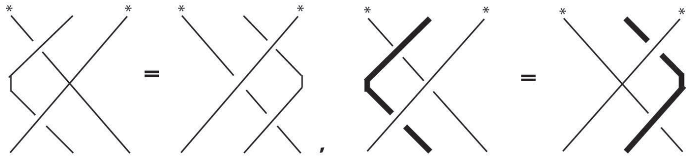


FIGURE 5.2. The colored Pachner $(2,3)$ moves, where the orientation of each strand is arbitrary, and the thickness of each $*$-marked strand is arbitrary

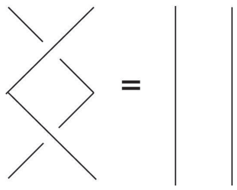


FIGURE 5.3. The colored $(0,2)$ moves, where the orientation and the thickness of each strand are arbitrary

Let $\mathcal{CD}$ be the set of colored diagrams. For $\mu = (\mu_1 \dots, \mu_n), \nu = (\nu_1, \dots, \nu_n) \in \{\pm\}^n$, we denote by

$$\mathcal{CD}(\mu; \nu) \subset \mathcal{CD}$$

the set of n-component colored diagrams $Z = Z_1 \cup \dots \cup Z_n$ such that

$$Z_i \text{ is thin } \Leftrightarrow \mu_i = +, \quad Z_i \text{ is thick } \Leftrightarrow \mu_i = -,$$
$$\partial Z_i \neq \emptyset \Leftrightarrow \nu_i = +, \quad \partial Z_i = \emptyset \Leftrightarrow \nu_i = -.$$

For $i = 1, \dots, n$, set

$$H(A)_i^+ = H(A), \quad H(A)_i^- = H(A)/N_{H(A)},$$
$$(H(A)^{\mathrm{op}})_i^+ = H(A)^{\mathrm{op}}, \quad (H(A)^{\mathrm{op}})_i^- = H(A)^{\mathrm{op}}/N_{H(A)^{\mathrm{op}}}.$$

We define the map

$$J'' \colon \mathcal{CD}(\mu; \nu) \to \bigotimes_{\mu_i=+} H(A)_i^{\nu_i} \bigotimes_{\mu_j=-} (H(A)^{\mathrm{op}})_j^{\nu_j}$$

in a similar way to the definition of J'' in Section 4, i.e., by putting the labels on the crossings as in Figure 4.3, not putting label for other fundamental tangle diagrams, and by taking the product of the labels.

5.2. **Colored moves.** We define several moves on colored diagrams, which we call *colored moves*, as follows.

The *colored Pachner* $(2,3)$ *moves* is the moves defined in Figure 5.2. We have $2 \times 2^2 \times 2^3$ kinds of colored Pachner $(2,3)$ moves.

We define the *colored* $(0,2)$ *moves* as the moves in Figure 5.3. We have $2^2 \times 2^2$ kinds of colored $(0,2)$ moves.

We define the *colored* $(0,8)$ *move* as the move in Figure 5.4.

We define the *symmetry moves* to be the moves in Figure 5.5.

It is known that if two tangle diagrams D and D' are isotopic to each other, then D and D' are related by a sequence of the moves defined in Figure 5.6, see e.g., [Kas95]. We define the *planar isotopies* of colored diagrams as the moves in Figure 5.6, where the thickness of each strand is arbitrary.

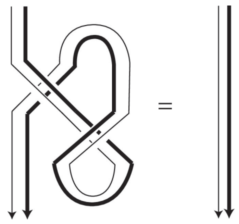


FIGURE 5.4. The colored $(0, 8)$ move

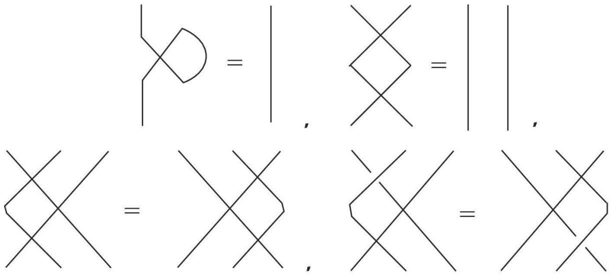


FIGURE 5.5. The symmetry moves, where the orientation and thickness of each strand are arbitrary

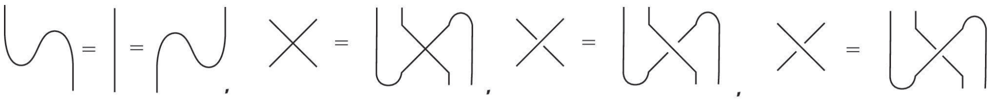


FIGURE 5.6. The planar isotopies, where the orientation of each strand is arbitrary

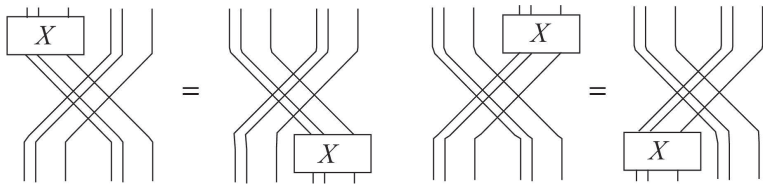


FIGURE 5.7. Naturality of the symmetry, where the orientations and the thickness of each strand are arbitrary. Here X is an arbitrary colored diagram.

Note that the family of moves in Figure 5.7, which is called the *naturality* of the symmetry, follows from the symmetry moves and planar isotopies.

Let $\sim_c$ be the equivalence relation on the set of colored diagrams generated by colored Pachner $(2, 3)$ moves, colored $(0, 2)$ moves, colored $(0, 8)$ moves, symmetry moves and planar isotopies.

Similarly, let $\sim'_c$ be the equivalence relation on the set of colored diagrams generated by colored Pachner $(2, 3)$ moves, colored $(0, 2)$ moves, colored $(0, 8)$ moves, symmetry moves and planar isotopies not involving a crossing,  and  at the same time.

We have the following.

Theorem 5.1. *The map J'' is an invariant under $\sim'_c$. If $\gamma^2 = 1$, then the map J'' is also an invariant under $\sim_c$.*

Proof. Let Z and Z' be two colored diagram.

If Z and Z' are related by a colored Pachner $(2,3)$ move, then $J''(Z) = J''(Z')$ follows from the pentagon relations (3.12)–(3.15).

If Z and Z' are related by a colored $(0,2)$ move, then $J''(Z) = J''(Z')$ follows from the invertibility of S, S', S'', and $\tilde{S}$.

If Z and Z' are the two colored diagrams in left and right hand sides, respectively, of the colored $(0,8)$ move, then we have $J''(Z) = J''(Z') = 1$.

If Z and Z' are related by a symmetry move, or by a planar isotopy which does not involve a crossing, then it is easy to see $J''(Z) = J''(Z')$.

If Z and Z' are related by a planar isotopy which involves a crossing but does not involve $\frown\!\searrow$ or $\smile\!\nearrow$, then $J''(Z) = J''(Z')$ follows from

$$\begin{aligned}
&(\gamma\otimes 1)(S) = S^{-1}, \quad (1\otimes\bar{\gamma}^*)(S^{-1}) = S, \quad (\gamma\otimes\bar{\gamma}^*)(S) = S \quad \in H(A)\otimes H(A),\\
&(\gamma\otimes 1)(S') = (S')^{-1}, \quad (1\otimes\bar{\gamma}^*)((S')^{-1}) = S', \quad (\gamma\otimes\bar{\gamma}^*)(S') = S' \quad \in H(A)^{\mathrm{op}}\otimes H(A),\\
&(\gamma\otimes 1)(S'') = (S'')^{-1}, \quad (1\otimes\bar{\gamma}^*)((S'')^{-1}) = S'', \quad (\gamma\otimes\bar{\gamma}^*)(S'') = S'' \quad \in H(A)\otimes H(A)^{\mathrm{op}},\\
&(\gamma\otimes 1)(\tilde{S}) = \tilde{S}^{-1}, \quad (1\otimes\bar{\gamma}^*)(\tilde{S}^{-1}) = \tilde{S}, \quad (\gamma\otimes\bar{\gamma}^*)(\tilde{S}) = \tilde{S} \quad \in H(A)^{\mathrm{op}}\otimes H(A)^{\mathrm{op}}.
\end{aligned}$$

If $\gamma^2 = 1$, then we also have

$$\begin{aligned}
&(1\otimes\bar{\gamma}^*)(S) = S^{-1}, \quad (\gamma\otimes 1)(S^{-1}) = S \quad \in H(A)\otimes H(A),\\
&(1\otimes\bar{\gamma}^*)(S') = (S')^{-1}, \quad (\gamma\otimes 1)((S')^{-1}) = S' \quad \in H(A)^{\mathrm{op}}\otimes H(A),\\
&(1\otimes\bar{\gamma}^*)(S'') = (S'')^{-1}, \quad (\gamma\otimes 1)((S'')^{-1}) = S'' \quad \in H(A)\otimes H(A)^{\mathrm{op}},\\
&(1\otimes\bar{\gamma}^*)(\tilde{S}) = \tilde{S}^{-1}, \quad (\gamma\otimes 1)(\tilde{S}^{-1}) = \tilde{S} \quad \in H(A)^{\mathrm{op}}\otimes H(A)^{\mathrm{op}},
\end{aligned}$$

which induce $J''(Z) = J''(Z')$ if Z and Z' are related by a planar isotopy involving a crossing, $\frown\!\searrow$ and $\smile\!\nearrow$ at the same time.

Thus we have the assertion. $\square$

5.3. **Tangles and colored diagrams.** Recall from Section 4.1 we consider the diagram $\zeta(D)$ associated to a tangle diagram D. Actually $\zeta(D)$ is nothing but a colored diagram and ζ defines a map giving a colored diagram from a tangle diagram. Recall also from (4.1) the map $(\leftarrow)$ giving a tangle diagram not involving $\frown\!\searrow$ or $\smile\!\nearrow$ from a tangle diagram. Actually, the composition

$$\zeta\circ(\leftarrow)\colon \{\text{tangle diagrams}\} \to \{\text{colored diagrams}\}/\sim_c' \quad (\to \{\text{colored diagrams}\}/\sim_c)$$

factors through $\{\text{tangle diagrams}\}/\sim_R$, where $\sim_R$ is the equivalence relation of tangle diagrams generated by Reidemeister I, II, III moves and planar isotopies of tangle diagrams. Thus $\zeta\circ(\leftarrow)$ is defined as the map from the isotopy class of non-framed tangles to the equivalence class of colored diagrams with respect to $\sim_c'$. Namely, we have the following.

Theorem 5.2. *Let D and D' be two diagrams of a tangle T. Then we have $\zeta(D_{(\leftarrow)}) \sim_c' \zeta(D'_{(\leftarrow)})$.*

Proof. Let D and D' be two tangle diagrams related by a Reidemeister II move. We can transform $\zeta(D_{(\leftarrow)})$ to $\zeta(D'_{(\leftarrow)})$ by applying colored $(0,2)$ moves four times, see Figure 5.8 for the case that each strand is oriented downwards.

Let D and D' be two tangle diagrams related by a Reidemeister III move. We can transform $\zeta(D_{(\leftarrow)})$ to $\zeta(D'_{(\leftarrow)})$ by applying colored Pachner $(2,3)$ moves eight times, see Figure 5.9 for the case that each strand is oriented downwards.

Recall from Figure 5.6 the planar isotopies of tangle diagrams.

Let D and D' be two tangle diagrams which are related by a planar isotopy in Figure 5.6 not involving $\frown\!\searrow$ and $\smile\!\nearrow$. We can transform $\zeta(D_{(\leftarrow)})$ and $\zeta(D'_{(\leftarrow)})$ by planar isotopies not involving $\frown\!\searrow$ and $\smile\!\nearrow$, see Figure 5.10 for examples.

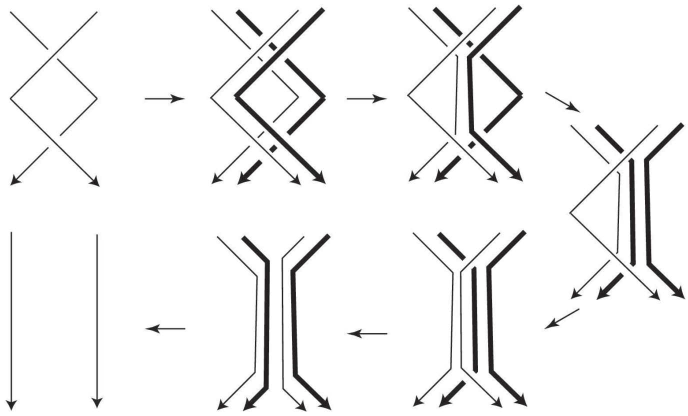


FIGURE 5.8. A realization of Reidemeister II move

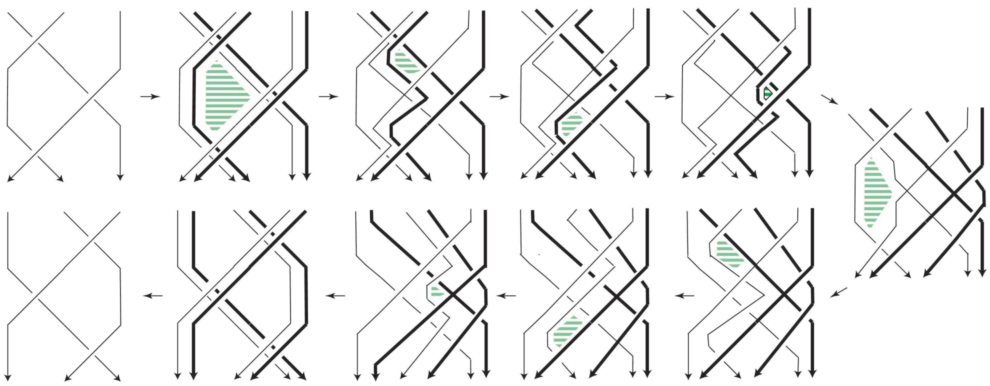


FIGURE 5.9. A realization of Reidemeister III move

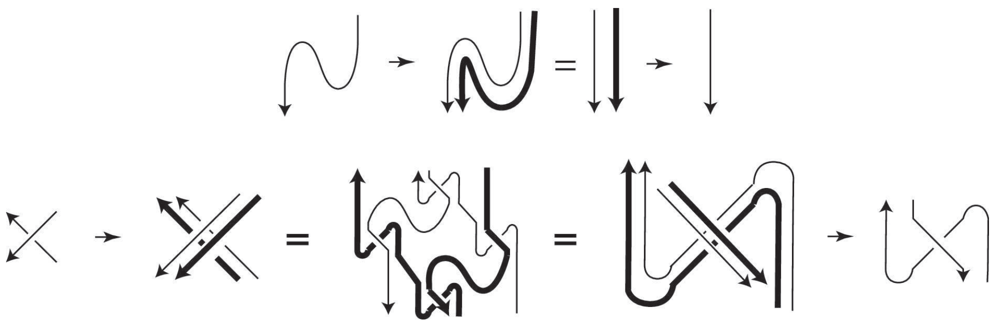


FIGURE 5.10. Realizations of planar isotopies of tangle diagrams not involving local maxima and minima going from left to right

Let D and D' be two tangle diagrams which are related by a planar isotopies move in Figure 5.6 involving $\frown$ or $\smile$. Note that using Reidemeister II and planar isotopies not involving $\frown$ and $\smile$, we have variants of Reidemeister II move as in Figure 5.11. We can transform $D_{(\leftarrow)}$ and $D'_{(\leftarrow)}$ by (variants of) Reidemeister II moves, Reidemeister III moves and planar isotopies not involving $\frown$ and $\smile$, see Figure 5.12 for examples.

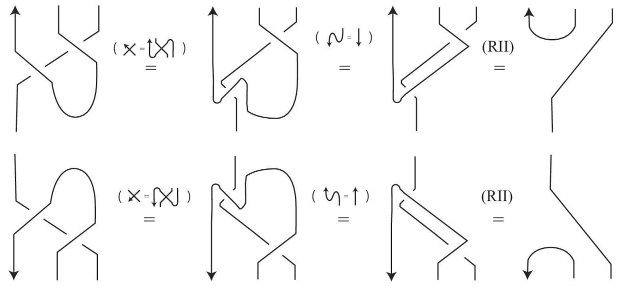


FIGURE 5.11. Variants of Reidemeister II moves

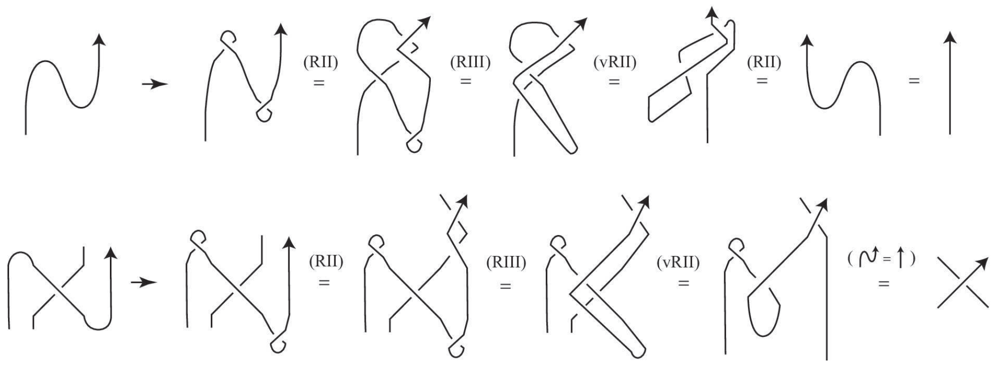


FIGURE 5.12. Realizations of planar isotopies of tangle diagrams involving local maxima and minima going from left to right

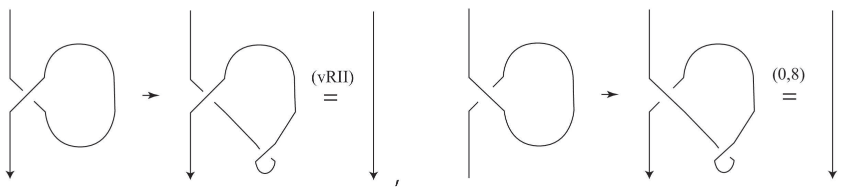


FIGURE 5.13. Realization of Reidemeister I move

Reidemeister III move	$\xrightarrow{J''}$	quantum Yang-Baxter equation
Colored Pachner $(2,3)$ moves	$\xrightarrow{J''}$	pentagon relations
Figure 5.9 (Colored $(2,3)$ move $\Rightarrow$ RIII move)	$\xrightarrow{J''}$	Lemma 3.6 (pentagon relation $\Rightarrow$ quantum Yang-Baxter equation)

TABLE 1. Correspondence between topological situation and algebraic situation

Let D and D' be two tangle diagrams related by a Reidemeister I move. We can transform $D_{(\leftarrow)}$ to obtain $D'_{(\leftarrow)}$ by the variant of Reidemeister II move and a colored $(0,8)$ move as in Figure 5.13.

$\square$

Note that the diagrammatic transformations in Figure 5.9 induces algebraic equations via the universal invariant J'', which gives a proof of Lemma 3.6. See the Table 1 for the situation.

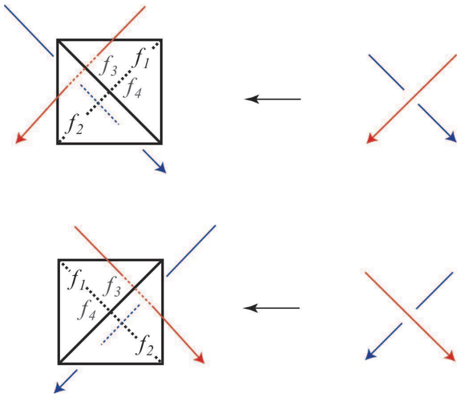


FIGURE 6.1. Two types of tetrahedra which are sticked by red and blue strands

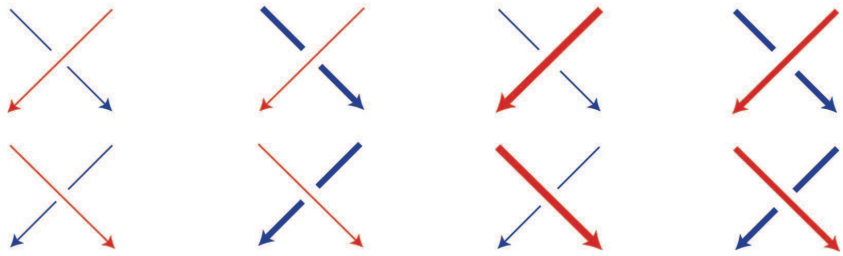


FIGURE 6.2. Colored tetrahedra

6. 3-dimensional descriptions; colored diagrams and colored singular triangulations

Recall from Section 5 that we extended the universal quantum invariant to an invariant of colored diagrams. In this section we associate a *colored tetrahedron* to each crossing of a colored diagram Z, and define a *colored cell complex* associated to Z. Using a colored cell complex we define a *colored singular triangulation* of a topological space. As a result, the universal quantum invariant J'' turns out to be an invariant of colored singular triangulations, where a copy of the S-tensor is attached to each colored tetrahedron.

6.1. **Colored tetrahedra.** Consider a tetrahedron Γ with an ordering of its 2-faces f_1, f_2, f_3, f_4. We stick Γ by a red (resp. blue) strand going into Γ at f_1 (resp. f_3) and out of Γ at f_2 (resp. f_4). Note that there are two types of such tetrahedra up to rotation as the left pictures in Figure 6.1. These tetrahedra are simply presented by crossings of red and blue strands as in the right pictures of Figure 6.1. We consider two types of strands, depicted by thick and thin strands, and then there are eight types of such sticked tetrahedra, which we call *colored tetrahedra*, presented by eight types of crossings with colors as in Figure 6.2.

6.2. **Colored diagrams and colored cell complexes.** We define a *colored cell complex* $\mathcal{C}(Z)$ associated to Z as follows.

Recall that a colored diagram Z consists of fundamental tangles and symmetries. Let $\{c_1, \dots c_k\}$ be the set of crossings in Z and $\{a_1, \dots, a_r\}$ the set of local maxima and minima, and boundary points of Z. To each crossing c_i, associate a colored tetrahedron Γ_i as in Section 6.1, by coloring (locally) the over-strand red and the

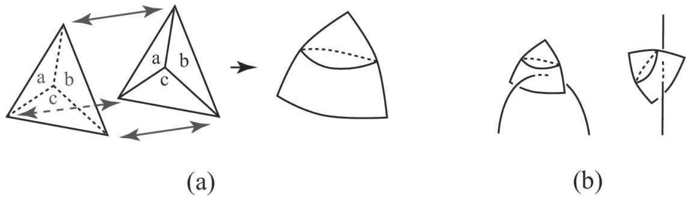


FIGURE 6.3. (a) A pillow, (b) How to place pillows

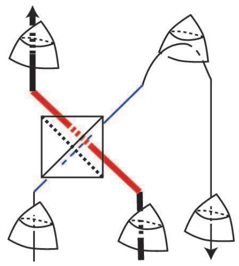


FIGURE 6.4. How to associate tetrahedra and pillows on a colored diagram

under-strand blue. A *pillow* is a 3-ball obtained from two tetrahedra by attaching them as in Figure 6.3 (a). To each a_i, we associate a pillow q_i as in Figure 6.3 (b). See Figure 6.4 for an example.

We define $\mathcal{C}(Z)$ to be the cell complex obtained from colored tetrahedra $\Gamma_1, \ldots, \Gamma_k$ and pillows $q_1, \ldots, q_r$ by gluing them along their faces as follows.

(1) 2-faces F and F' of $\Gamma_1, \ldots, \Gamma_k$ or $q_1, \ldots, q_r$ are glued if and only if F and F' are adjacent along Z.
(2) We define the *red vertex* and the *blue vertex* of each 2-face of $\Gamma_1, \ldots, \Gamma_k$ and $q_1, \ldots, q_r$ as in Figure 6.5, and glue adjacent faces F and F' so that the vertices of the same color are attached.

More precisely, the red and the blue vertices are defined as follows.

Consider a crossing c_i of Z, with the associated tetrahedron Γ_i. The *red faces* of Γ_i are the two faces of Γ_i that are pierced by the red strand, i.e., by the over-strand. The *red vertex* of a red face F of Γ_i is the vertex of F that is not shared with the other red face of Γ_i. We define *blue faces* of Γ_i and the *blue vertex* of each blue face of Γ_i similarly. On each 2-face F of Γ_i, we define the cyclic order among its three vertices by right hand screw rule with the strand piercing F as a thumb. The *blue vertex* of a red face and the *red vertex* of a blue face are defined so that

(A) if the 2-face are pierced by thin (resp. thick) strand, then the red vertex is the next (resp. previous) vertex of the blue vertex.

Consider a local maximum, a local minimum, or a boundary point a_j of Z, with the associated pillow q_j. We define the *red vertex* and the *blue vertex* of each face of q_j so that the above condition (A) holds and that the vertices of the same color in two 2-faces of q_j are compatible (attached) in q_j.

6.3. **Colored singular triangulations and colored ideal triangulations.** For a space X, a *singular triangulation* (see e.g., [TV92, BB04]) of X consists of a finite index set I, a function $d\colon I \to \mathbb{N}$, and continuous maps $f_i\colon \Delta^{d(i)} \to X$ for $i \in I$, where Δ^n is the standard n simplex, such that $(I, d, \{f_i\}_{i\in I})$ is a finite cell decomposition of X, and for each $i \in I$ and a face F in $\Delta^{d(i)}$, the restriction $f_i|_F$

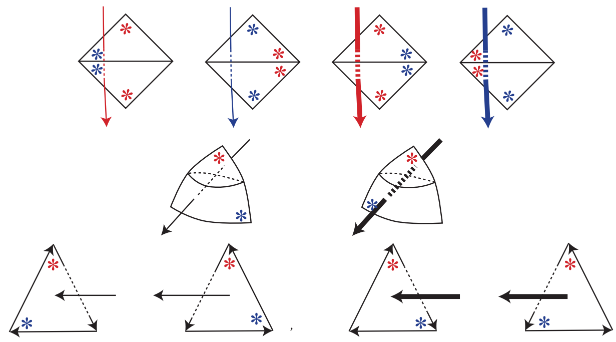


FIGURE 6.5. How to color two distinct vertices of a triangle red and blue

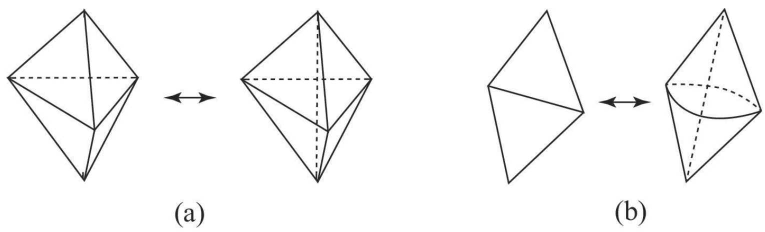


FIGURE 6.6. (a) The Pachner $(2,3)$ move, (b) The $(0,2)$ move

is the composition $f_j \circ g$ of an affine isomorphism $g\colon F \to \Delta^{d(j)}$ and f_j for some $j \in I$.

Let $\mathcal{C}(Z)$ be the colored cell complex of a colored diagram Z, which we can naturally regard a singular triangulation. Consider

$$X = \mathcal{C}(Z)/(e_1 = e_1', \dots, e_k = e_k', v_1 = v_1', \dots, v_l = v_l')$$

be a singular triangulation obtained from $\mathcal{C}(Z)$ by identifying some pairs of edges $(e_1, e_i'), \dots, (e_k, e_k')$, $k \geq 0$, and some pairs of vertices $(v_1, v_1'), \dots, (v_l, v_l')$, $l \geq 0$, in $\mathcal{C}(Z)$. Let $\psi\colon \mathcal{C}(Z) \to X$ be the projection. We call the triple (X, Z, ψ) a *colored singular triangulation of type* Z. In particular, if X is an ideal triangulation of some topological space $\tilde{X}$, then we call it a *colored ideal triangulation of* $\tilde{X}$.

Let $\mathcal{CT}(Z)$ be the set of colored singular triangulations of type Z and set

$$\mathcal{CT} = \bigcup_{Z \in \mathcal{CD}} \mathcal{CT}(Z).$$

6.4. **Colored moves and colored singular triangulations.** We can translate colored moves on the set $\mathcal{CD}$ of colored diagrams defined in Section 5.2 to moves on the set of colored cell complexes as follows.

Let Z and Z' be two colored diagrams. If Z and Z' are related by a colored Pachner $(2,3)$ move, then $\mathcal{C}(Z)$ and $\mathcal{C}(Z')$ are related by a Pachner $(2,3)$ move defined as in Figure 6.6 (a), which is a move replacing two tetrahedra sharing one face with three tetrahedra, or its inverse. See Figure 6.7 for an example. Here, note that the values $J''(Z)$ and $J''(Z')$ of the universal quantum invariant are related by a pentagon relation.

If Z and Z' are related a colored $(0,2)$ move, then $\mathcal{C}(Z')$ is obtained from $\mathcal{C}(Z)$ by the move (or its inverse) first attaching two faces F and F' of $\mathcal{C}(Z)$ along the edges e of F and e' of F', which is determined depending on the color and thickness of the strand piercing F and F', and then gluing two tetrahedra along F and F'.

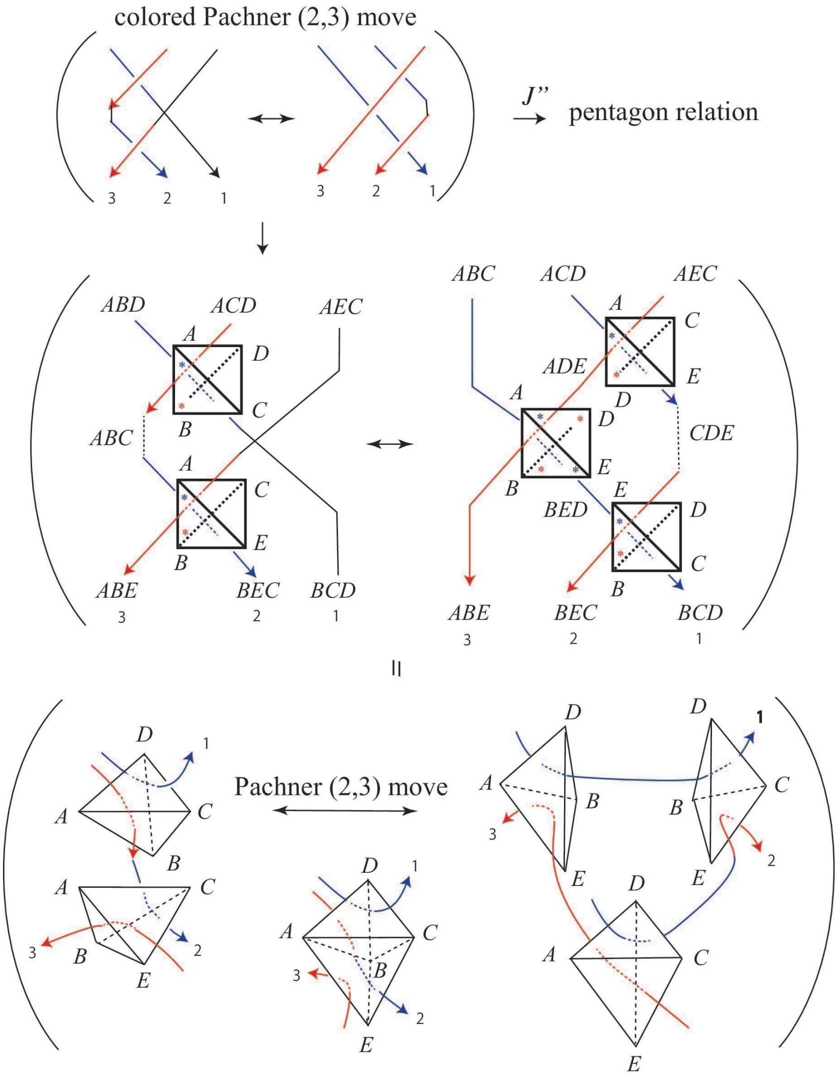


FIGURE 6.7. A colored Pachner $(2,3)$ move of colored diagrams and a Pachner $(2,3)$ move of colored cell complexes, whose image of J'' turns out to be the pentagon relation $S_{23}S_{12} = S_{12}S_{13}S_{23}$

If the edges e and e' are already attached in $\mathcal{C}(Z)$, then the colored $(0,2)$ move is nothing but a $(0,2)$ move as in Figure 6.6 (b), which is a move replacing two adjacent 2-faces with two tetrahedra, or its inverse.

If Z and Z' are related by a colored $(0,8)$ move, then $\mathcal{C}(Z')$ is obtained from $\mathcal{C}(Z)$ by removing or adding the eight tetrahedra of $\mathcal{C}(Z)$.

If Z and Z' are related by a planar isotopy, then $\mathcal{C}(Z)$ and $\mathcal{C}(Z')$ are isomorphic to each other.

Let (X, Z, ψ) and (X', Z', ψ') be colored singular triangulations. We say that (X, Z, ψ) and (X', Z', ψ') are related by a colored Pachner $(2,3)$ move if

(1) the colored diagram Z and Z' are related by a colored Pachner $(2,3)$ move, and
(2) $\psi = \psi'$ on the exteriors $\mathcal{C}(Z) \setminus W = \mathcal{C}(Z') \setminus W'$, where W (resp. W') is the subcomplexes of $\mathcal{C}(Z)$ (resp. $\mathcal{C}(Z')$) consisting of the three (resp. two)

tetrahedra corresponding to the three (resp. two) crossings of Z (resp. Z') involved in the colored Pachner $(2,3)$ move.

We define other colored moves on colored singular triangulations similarly.

Similarly to the equivalence relations $\sim_c$ and $\sim'_c$ on $\mathcal{CD}$, let $\sim_{ct}$ be the equivalence relation on $\mathcal{CT}$ generated by colored Pachner $(2,3)$ moves, colored $(0,2)$ moves, colored $(0,8)$ moves, symmetry moves and planar isotopies, and $\sim'_{ct}$ the equivalence relation on $\mathcal{CT}$ generated by colored Pachner $(2,3)$ moves, colored $(0,2)$ moves, colored $(0,8)$ moves, symmetry moves and planar isotopies not involving a crossing, and at the same time.

Let

$$\pi\colon\ \mathcal{CT} \to \mathcal{CD},$$

be the map such that $\pi\left((X,Z,\psi)\right) = Z$ for $(X,Z,\psi) \in \mathcal{CT}(Z)$.

For $\mu = (\mu_1,\dots,\mu_m), \nu = (\nu_1,\dots,\nu_m) \in \{\pm\}^m$, $m \geq 0$, let $\mathcal{CT}(\mu;\nu)$ be the set of colored singular triangulations of types in $\mathcal{CD}(\mu;\nu)$.

Proposition 6.1. *The composition*

$$J'' \circ \pi\colon\ \mathcal{CT}(\mu;\nu) \to \bigotimes_{i\in I_+} H(A)_i^{\nu_i} \bigotimes_{j\in I_-} (H(A)^{\mathrm{op}})_j^{\nu_j}$$

of the restriction of π to $\mathcal{CT}(\mu;\nu)$ and the universal quantum invariant J'' is an invariant under $\sim'_{ct}$. If $\gamma^2 = 1$, then $J''\circ\pi$ is also an invariant under $\sim_{ct}$.

Proof. Note that the projection map π induces the map

$$\mathcal{CT}/\sim'_{ct} \to \mathcal{CD}/\sim'_c \quad (\text{resp. } \ \mathcal{CT}/\sim_{ct} \to \mathcal{CD}/\sim_c),$$

which shows the invariance of $J''\circ\pi$ under $\sim'_{ct}$ (resp. $\sim_{ct}$ if $\gamma^2 = 1$). □

We call $J''\circ\pi$ the *universal quantum invariant* of colored singular triangulations.

6.5. **Colored diagrams and colored 2-complexes.** An (m,n)-*colored diagram* is a colored diagram with m boundary points on the top and n boundary points on the bottom.

An m-*colored* 2-*complex* is a singular triangulated 2-complex with m ordered triangles, such that two distinct vertices of each triangle are marked by red and the blue.

Let Z be an (m,n)-colored diagram and $\mathcal{C}(Z)$ the colored cell complex associated to Z. Recall that in the construction of $\mathcal{C}(Z)$ we associate a pillow to each boundary points of Z. We denote by $s_t(Z)$ (resp. $s_b(Z)$) the triangulated subsurface in $\mathcal{C}(Z)$ consisting of the triangles associated to the boundary points on the top (resp. bottom) of Z. Note that $s_t(Z)$ (resp. $s_b(Z)$) is a colored 2-complex with the order induced by the order of the boundary points on the top (resp. bottom), and the red and the blue marks induced by the rule in Section 6.2, see also Figure 6.5, depending on the orientation and the thickness of each strand at the top (resp. bottom) of Z.

For an (m,n)-colored diagram Z and an (m',n')-colored diagram Z', let $Z\otimes Z'$ be the an $(m+m', n+n')$-colored diagram obtained by attaching Z to the left of Z'. For an (n,l)-colored diagram W and an (m,n)-colored diagram W', let $W\circ W'$ be the (m,l)-colored diagram obtained by attaching W to the bottom of W'.

For an m-colored 2-complex X and m'-colored 2-complex X', we define the $(m+m')$-colored 2-complex $X\cup X'$ as the disjoint union of X and X' such that the order of each triangle of X' is increased by m. Note that

$$s_t(Z\otimes Z') = s_t(Z)\cup s_t(Z'), \quad s_b(Z\otimes Z') = s_b(Z)\cup s_b(Z'),$$
$$s_t(W\circ W') = s_t(W'), \quad s_b(W\circ W') = s_b(W).$$

6.6. **Transformations of colored 2-complexes.** Let Z be an (m,n)-colored diagram and X an m-colored 2-complex. Let Y be the m-colored 2-complex obtained from X by attaching $s_t(Z)$ so that for $i = 1, \dots, m$, the red and blue vertices of the ith triangle of X is attached to the red and blue vertices, respectively, of the ith triangle of $s_t(Z)$.

Let $i_1\colon\ s_t(Z) \to Y$ and $i_2\colon\ s_t(Z) \to \mathcal{C}(Z)$ be the inclusion maps. We define the n-colored 2-complex $\rho(Z)(X)$ as the image of $s_b(Z)$ in the pushout W of i_1 and i_2;

$$\begin{tikzcd} s_t(Z) \arrow[r, "i_1"'] \arrow[dr, "i_2"'] & Y \arrow[dr] & \\ s_b(Z) \arrow[r] & \mathcal{C}(Z) \arrow[r] & W \end{tikzcd}$$

By the definition we have $\rho(Z)(s_t(Z)) = s_b(Z)$. Thus we obtain the map

$$\rho(Z)\colon \{m\text{-colored 2-complexes}\} \to \{n\text{-colored 2-complexes}\} \quad X \mapsto \rho(Z)(X).$$

For (m,n)- and (m',n')-colored diagrams Z and Z', respectively, and m- and m'-colored 2-complex X and X', respectively, set

$$(\rho(Z) \otimes \rho(Z'))(X \cup X') = \rho(Z)(X) \cup \rho(Z')(X').$$

We have the following.

Proposition 6.2. (1) *For a vertical trivial strand t, we have $\rho(t) = \mathrm{id}$.*

(2) *For a symmetry s, the map $\rho(s)$ is the transposition of the order of the two triangles.*

(3) *Let c^+ be a crossing with over-strand connecting the right top boundary point to the left bottom boundary point. Then the map $\rho(c^+)$ is defined in Figure 6.8(a).*

(4) *Let c^- be a crossing with over-strand connecting the right top boundary point to the left bottom boundary point. Then the map $\rho(c^-)$ is defined in Figure 6.8(b).*

(5) *For a local maximum $\cap$, the map $\rho(\cap)$ is defined in 6.9 (a).*

(6) *For a local minimum $\cup$, the map $\rho(\cup)$ is defined in 6.9 (b), where two edges e and e' with the same arrow in the left picture are attached in the right picture. If both of e and e' are not shared by any face in the left picture, then there is no edge with the same arrow in the right picture.*

(7) *For colored diagrams Z and Z', We have $\rho(Z \otimes Z') = \rho(Z) \otimes \rho(Z')$.*

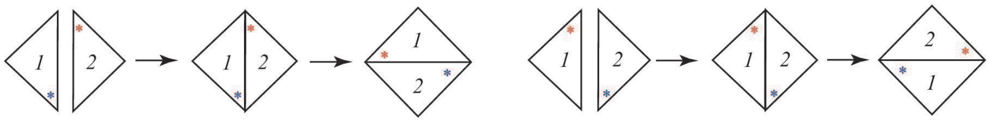
(8) *For an (n,l)-colored diagram W and an (m,n)-colored diagram W', we have $\rho(W \circ W') = \rho(W) \circ \rho(W')$.*

Proof. The colored cell complex $\mathcal{C}(t)$ of the vertical trivial strand t is obtained from two pillows p_1 and p_2 associated to the top and the bottom boundary points of t. Attaching a pillow does not change colored 2-complexes, and thus attaching $\mathcal{C}(t)$ neither. Thus we have (1).

Similarly, the colored cell complex $\mathcal{C}(s)$ of the symmetry s is obtained from four pillows p_1, p_2, p_3, p_4, where p_1 and p_2 are associated to top left and bottom right boundary points, respectively, and p_3 and p_4 are attached to top right and bottom left boundary points, respectively. Thus attaching $\mathcal{C}(s)$ exchanges the order of two triangulations in colored 2-complexes. Thus we have (2).

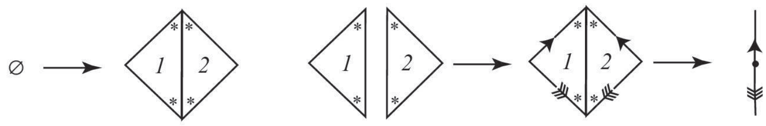
The colored cell complex $\mathcal{C}(c^\pm)$ of the crossing $c^\pm$ is a tetrahedron. Thus attaching $\mathcal{C}(c^\pm)$ induces the transformation gluing two triangulation F and F' following the marks on F and F', and applying a flip. Thus we have (3) and (4).

The colored cell complex $\mathcal{C}(\cap)$ of $\cap$ is a pillow obtained from three pillows by attaching two pillows along the two faces of the other one. Thus attaching $\mathcal{C}(\cap)$ induces creating two triangles sharing an edge, on colored 2-complexes. Oppositely,



(a) (b)

FIGURE 6.8. The maps (a) $\rho(c^+)$ and (b) $\rho(c^-)$, where the ambiguities of the place of the red or the blue mark in each triangle should be fixed following the rule defined in Section 6.2 depending on the orientation and the thickness of the corresponding strand of the crossings in $c^\pm$.



(a) (b)

FIGURE 6.9. The maps (a) $\rho(\cap)$ and (b) $\rho(\cup)$, where the ambiguities of the color of each mark should be fixed following the rule defined in Section 6.2 depending on the orientation and the thickness of $\cap$ and $\cup$, respectively.

attaching the colored cell complex $\mathcal{C}(\cup)$ of $\cup$ induces removing two triangles (and gluing corresponding edges adjacent to the triangles) on colored 2-complexes. Thus we have (5) and (6).

Let Z be an (m, n)-colored diagram and Z' be an (m', n')-colored diagram. The colored diagram $\mathcal{C}(Z \otimes Z')$ of $Z \otimes Z'$ is the disjoint union of $\mathcal{C}(Z)$ and $\mathcal{C}(Z')$, and the colored 2-complexes $s_t(Z \otimes Z')$ (resp. $s_b(Z \otimes Z')$) is the disjoint union of $s_t(Z)$ and $s_t(Z')$ (resp. $s_b(Z)$ and $s_b(Z')$), where the order i of the surface in $s_t(Z')$ (resp. $s_b(Z')$) becomes $i + m$ in $s_t(Z \otimes Z')$ (resp. $i + n$ in $s_b(Z \otimes Z')$). Thus we have (7).

We can construct $\mathcal{C}(W \circ W')$ by gluing the top surface of $\mathcal{C}(W)$ to the bottom surface of $\mathcal{C}(W')$, which induces the decomposition $\rho(W \circ W') = \rho(W) \circ \rho(W')$. Thus we have (8), which completes the proof. □

For an example, for the colored diagram $\zeta(c)$ of a positive crossing c, the transformation $\rho(\zeta(c))$ of colored 2-complexes is depicted as in Figure 6.10, where the each double edge presents two distinct edges with four distinct vertices in the colored cell complex $\mathcal{C}(\zeta(c))$ associated to $\zeta(c)$ (which will be identified in the octahedral triangulation $\mathcal{O}(c)$ associated to c defined in the following sections). We remark that Figure 6.10 without color is observed in several papers on mapping class groups and on cluster algebras, see e.g., [Kas01, HI15].

Remark 6.3. Colored diagrams form a strict symmetric monoidal category $\mathcal{CD}/\sim_c$ as follows. The objects are sequences of $\downarrow, \uparrow, \Downarrow, \Uparrow$, including an empty sequence $\emptyset$. For an object I, the identity id_I is the colored diagram in $[0, 1]^2$ consisting of vertical trivial strands (intervals connecting points on the top and on the bottom), where the thickness and the orientation of each strand follows these of each arrow in I. The composition and the tensor product are given by $\circ$ and $\otimes$ defined in Section

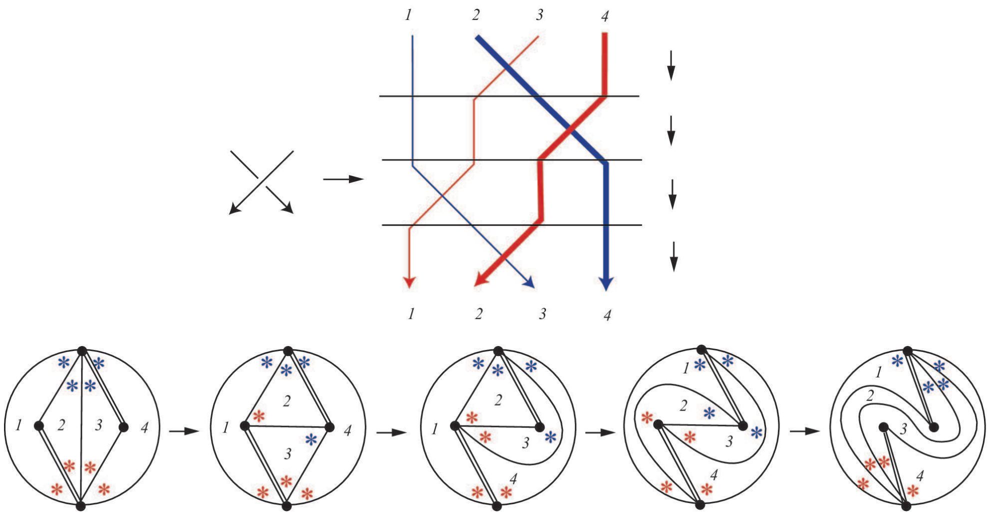


FIGURE 6.10. The transformation $\rho(\zeta(c))$ associated to the colored diagram $\zeta(c)$ of a positive crossing c.

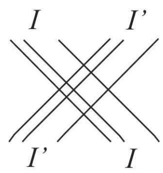


FIGURE 6.11. The symmetry

6.6. The tensor product $I \otimes I'$ of two objects I and I' is the sequence obtained by placing I' after I. The symmetry of the tensor product $I \otimes I'$ is given in Figure 6.11. Then, the category $\mathcal{CD}/\sim_c$ is generated as a strict symmetric monoidal category by the fundamental tangle diagrams and the equivalence relation $\sim_c$. Similarly, the category $\mathcal{CD}/\sim_c'$ is generated as a strict symmetric monoidal category by the fundamental tangle diagrams and the equivalence relation $\sim_c'$.

Note that by Theorem 5.2, there is a functor F from the category $\mathcal{T}$ of (non-framed) tangles to $\mathcal{CD}/\sim_c'$, and we can generalize the universal quantum invariant to a functor J'' from $\mathcal{CD}/\sim_c'$ so that $J'' \circ F$ is the universal quantum invariant (with a framing adjustment) for tangles (e.g., [KR01]).

7. Octahedral triangulation of tangle complements

In this section we define the ideal triangulations and the octahedral triangulations of tangle complements.

7.1. **Ideal triangulations of tangle complements.** Let M be a compact manifold of dimension $n \leq 3$, possibly with non-empty boundary. Let F be an $(n-1)$-submanifold of ∂M. Let $F_1, ..., F_k$ be the connected components of F. Let $M//F$ denote the topological space obtained from M by collapsing each F_i into a point. An *ideal triangulation* of the pair (M, F) is defined to be a singular triangulation of $M//F$ such that each vertex of the singular triangulation is on a point arising from F.

Let $D_n = [0,1]^2 \setminus (P_1 \cup \cdots \cup P_n)$ be a punctured disk, where $P_1, \ldots, P_n$ are small disks with the centers arranged on the line $[0,1] \times \{1/2\}$ as in Figure 7.1(a).

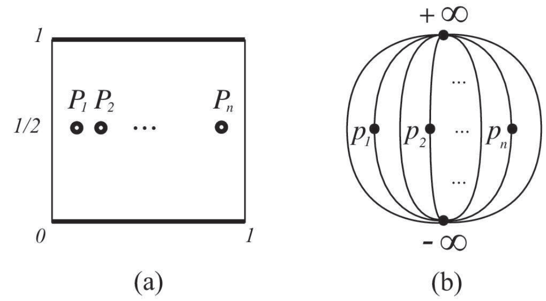


FIGURE 7.1. (a) A punctured disk and (b) its leaves-ideal triangulation

We define the *leaves-ideal triangulation* l_n of D_n to be the ideal triangulation of the pair $(D_n, ([0,1] \times \{0,1\}) \cup \partial P_1 \cup \cdots \cup \partial P_n)$ as in Figure 7.1(b), where we denote by $-\infty, +\infty, p_1, \ldots, p_n$ the vertices corresponding to $[0,1] \times \{0\}, [0,1] \times \{1\}, \partial P_1, \ldots, \partial P_n$, respectively. Here we formally define l_0 as a segment having $\{\pm\infty\}$ as its vertices. In particular we call l_1 a *leaf.*

Let $T = T_1 \cup \cdots \cup T_n$ be an n-component tangle. Let $E = \overline{[0,1]^3 \setminus N(T)}$ be the complement of T in the cube, where $N(T)$ is a tubular neighborhood of T in the cube. Let F_T be the intersection $\partial E \cap N(T)$, which consists of annuli and tori. Then an *ideal triangulation of the tangle complement* E of T in the cube is defined to be an ideal triangulation of $(E, F_T \cup F_{z=0} \cup F_{z=1})$, where $F_{z=0} = [0,1] \times [0,1] \times \{0\}$ and $F_{z=1} = [0,1] \times [0,1] \times \{1\}$, such that its restriction to each boundary component $[0,1] \times \{0,1\} \times [0,1]$ is a leaves-ideal triangulation. The vertices corresponding to $F_{z=0}$, and $F_{z=1}$ are denoted by $-\infty$ and $+\infty$, respectively.

7.2. **Colored ideal triangulations for octahedral triangulations of tangle complements.** Let T be a tangle and D its diagram. We define a cell complex $\mathcal{O}(D)$, which we call the *octahedral triangulations* associated to D, as follows.

Step 1. Take a colored diagram

Recall from Section 4 the colored diagram $\zeta(D)$ of D. On a duplicated crossing we color over-strand red, and under-strand blue. [4]

Step 2. Preparing and placing octahedra and double pillows

(i) Let $\{c_1, \ldots c_k\}$ be the set of crossings of the diagram D. In a neighborhood of crossing $\zeta(c_i)$, let t^i_1 be the right crossing with strings oriented downwards, and let t^i_2, t^i_3, t^i_4 other crossings, one by one in a counterclockwise order, see Fig 7.2. As in Figure 7.2, for $j = 1, 2, 3, 4$, we associate a tetrahedron $\Lambda^i_j = n^i_j \tilde{e}^i_j \tilde{e}'^i_j s^i_j$ to each t^i_j. Then we glue the four tetrahedra $\Lambda^i_1, \Lambda^i_2, \Lambda^i_3, \Lambda^i_4$ together to obtain an octahedron $o_i = n^i e^i_{12} e^i_{23} e^i_{34} e^i_{41} s^i$, so that n^i_j, $\tilde{e}^i_j$, $\tilde{e}'^i_j$, and s^i_j are going to n^i, $e^i_{j-1,j}$, $e^i_{j,j+1}$, and s^i, respectively, where the index j should be considered modulo 4. We place o_i between the two original strands of c_i so that n^i and s^i are placed on the over-strand and the under-strand, respectively.

(ii) Let $\{a_1, \ldots, a_r\}$ be the set of local maxima and minima, and boundary points of D. For $i = 1, \ldots, r$ we prepare a *double pillow* Q_i obtained from two pillows q_i and q'_i attaching to each other along two edges as in Figure 7.3 (a), and place Q_i to a_i as in Figure 7.3 (b).

Step 3. Gluing octahedra and double pillows

We glue the octahedra $o_1, \ldots, o_k$ and the double pillows $Q_1, \ldots, Q_r$ as follows.

[4]This process is not necessary since we can recover the color from a non-colored diagram.

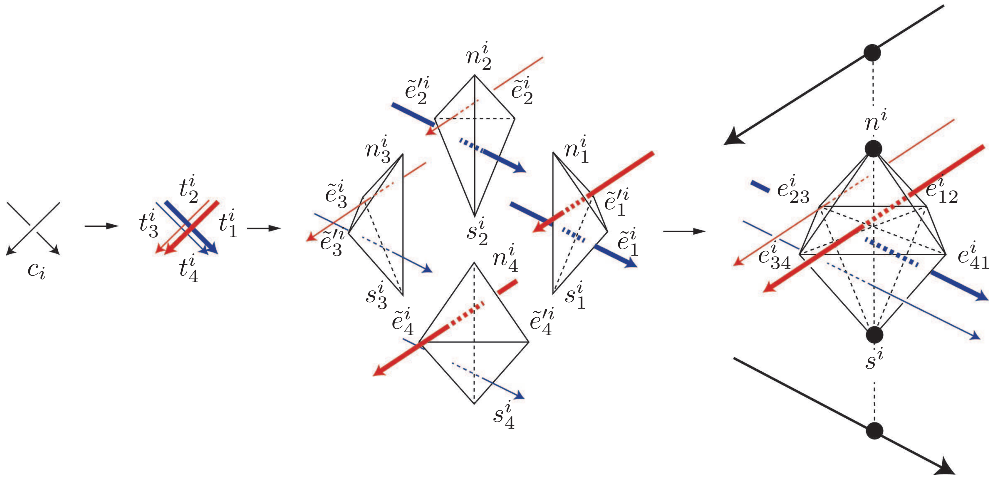
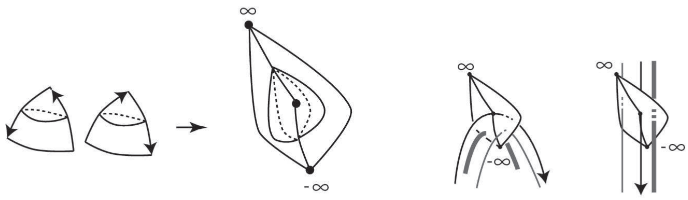


FIGURE 7.2. Octahedral triangulation around a crossing



(a) (b)

FIGURE 7.3. (a) A double pillow, (b) How to place a double pillow

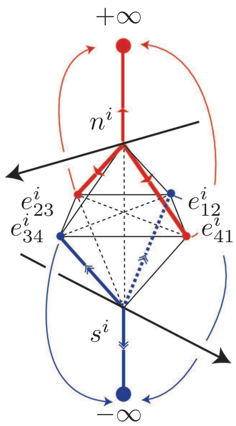


FIGURE 7.4. How to glue the edges in a octahedron

For each positive (resp. negative) crossing c_i, we pull the vertices e_{23}^i and e_{41}^i (resp. e_{12}^i and e_{34}^i) upwards, put them on $+\infty$, and glue the two edges n^i-e_{23}^i and n^i-e_{41} (resp. n^i-e_{12}^i and n^i-e_{34}). Similarly, pull the vertices e_{12}^i and e_{34}^i (resp. e_{23}^i and e_{41}^i) downwards, put them on $-\infty$, and glue the two edges s^i-e_{12}^i and s^i-e_{34} (resp. s^i-e_{23}^i and s^i-e_{41}), see Figure 7.4. Note that the boundary of the octahedron o_i consists of four leaves corresponding to the four edge of c_i, see Figure 7.5. Note also that the boundary of the double pillow Q_j consists of two leaves. We glue the octahedra $o_1, \dots, o_k$ and the double pillows $Q_1, \dots Q_r$ along the pairs of leaves which are adjacent on D so that $\pm\infty$ are attached compatibly.

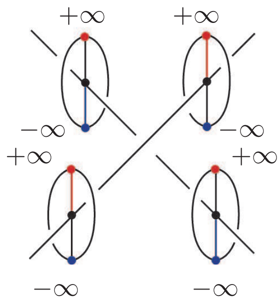


FIGURE 7.5. Leaves corresponding to the four edges of a crossing

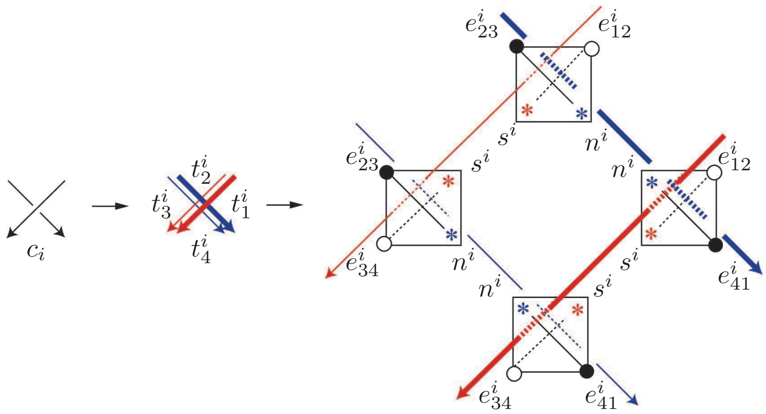


FIGURE 7.6. The colored ideal triangulation and the octahedron at a crossing of a tangle

Remark 7.1. The diagram D is called *splitting* if

(1) the 4-regular plane graph giving the diagram D is not connected, or
(2) there is a component of D such that crossings along the path of the component are only over-passing or only under-passing.

If D is not splitting and not isotopic to S^1 in $[0, 1]^2$, then $\mathcal{O}(D)$ is an ideal triangulation of the tangle complement E. If in addition D is a link diagram, then $\mathcal{O}(D)$ is nothing but the octahedral triangulation studied in e.g., [CKK14, Yok11] in the context of the hyperbolic geometry. If D is splitting or isotopic to S^1 in $[0, 1]^2$, then $\mathcal{O}(D)$ is not a 3-manifold.

We have the following.

Proposition 7.2. *The octahedral triangulation $\mathcal{O}(D)$ associated to a tangle diagram D admits a colored ideal triangulation of type $\zeta(D)$.*

Proof. In Step 2, we associate an octahedron o_i to each crossing c_i, where the octahedron is obtained from four tetrahedra as in Figure 7.2. Actually we can obtain o_i also from the colored diagram $\zeta(c_i)$ as depicted in Figure 7.6. We also associate the pillows q_j and q'^j to the two components of $\zeta(a_j)$. In Step 3, we glued the octahedra and double pillows as in Figure 7.7 (we omit to draw the pillows), which follows the gluing rule of the colored tetrahedra and pillows defined in Section 6.2. This completes the proof.

□

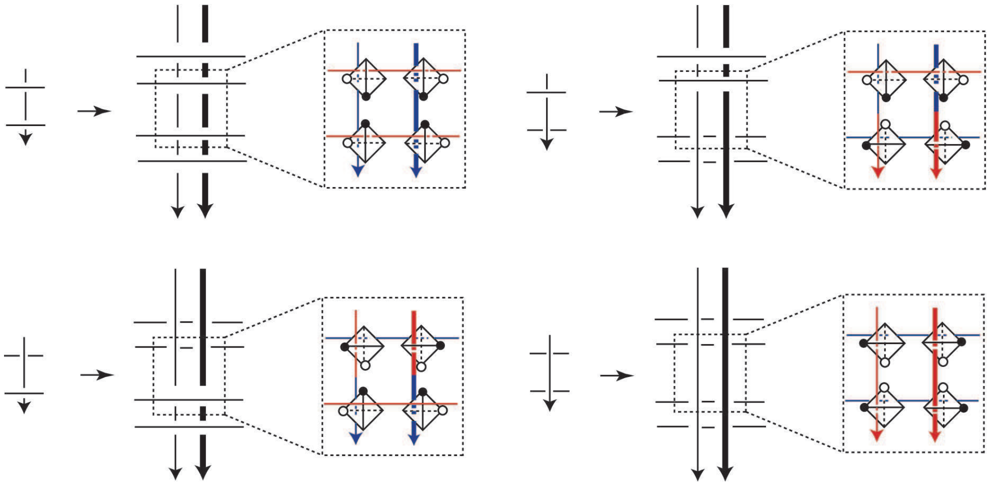


FIGURE 7.7. How we glued the octahedra in the octahedral triangulations, where the black dots are attached to $+\infty$ and the white dots are attached to $-\infty$.

Remark 7.3. A tangle complement could admit more than one colored ideal triangulations up to the equivalence relation $\sim'_{ct}$, and the universal quantum invariant could give different values on them. We expect that *the universal quantum invariant is an invariant of pairs of tangle complements and some geometrical inputs obtained from the color*, which we will study in [KST].

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